Palindromes in Finite Groups

B.Sc. Project in Mathematics

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- Question: How many positions, $f^*(n)$, are visited if both play optimally?

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- Round 1

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- Conclusion: $f^*(2) = 2$

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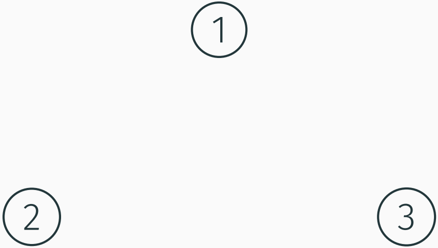

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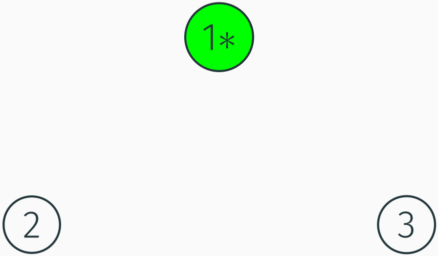

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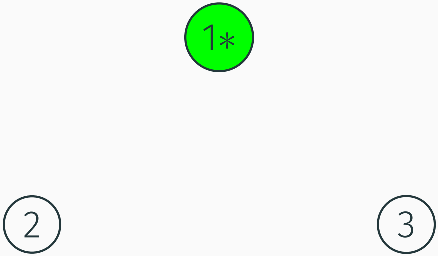

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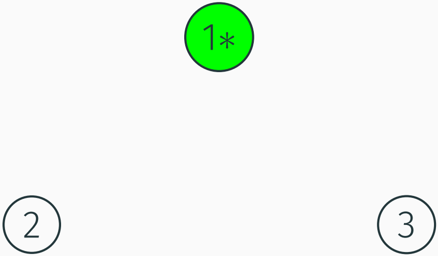

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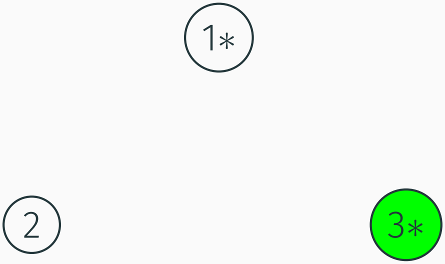

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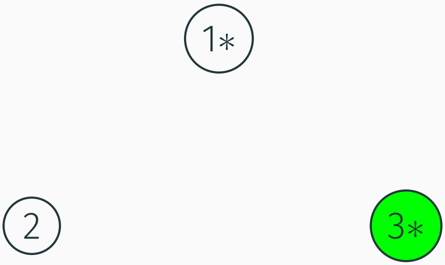

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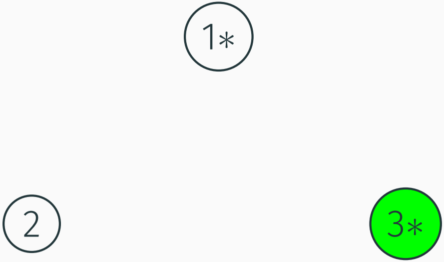

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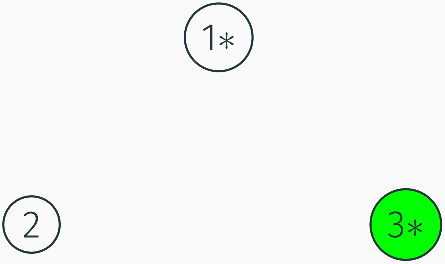

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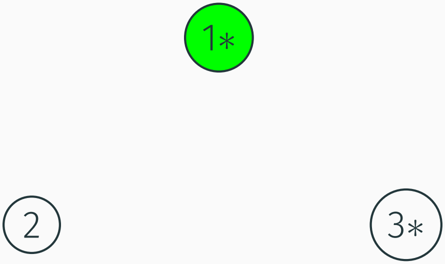

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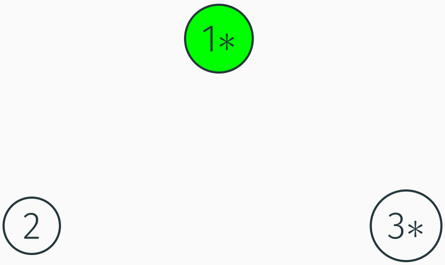

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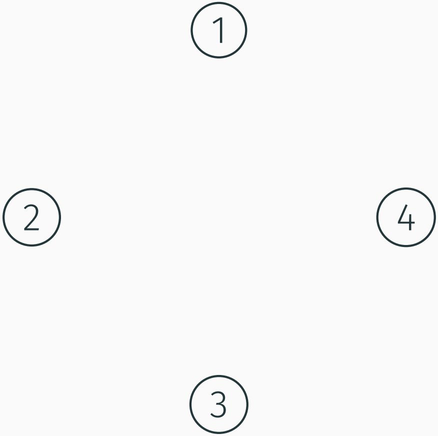

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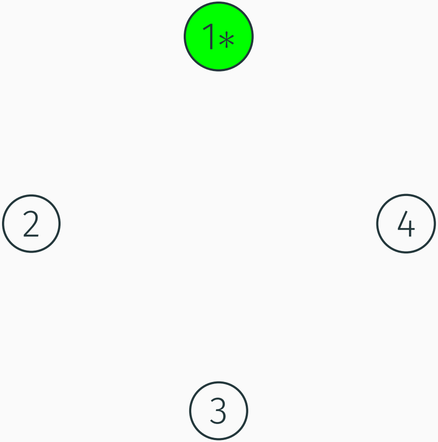

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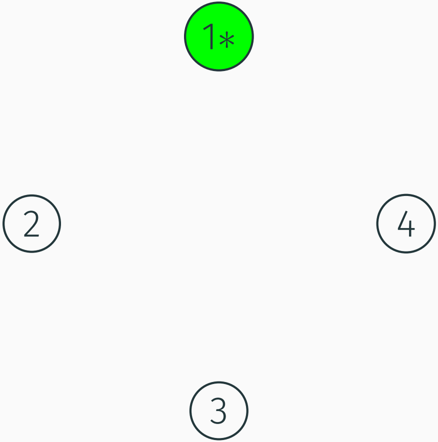

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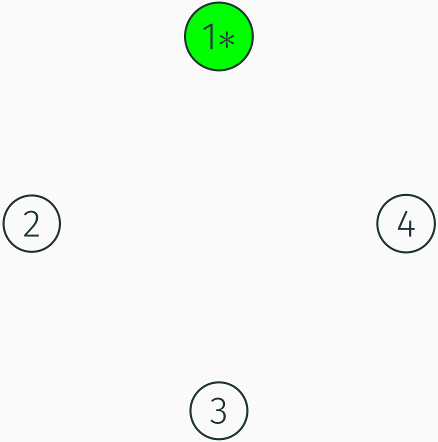

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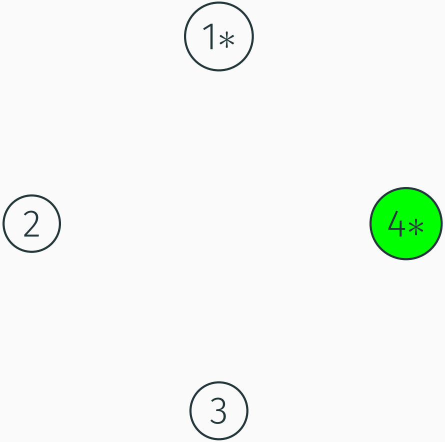

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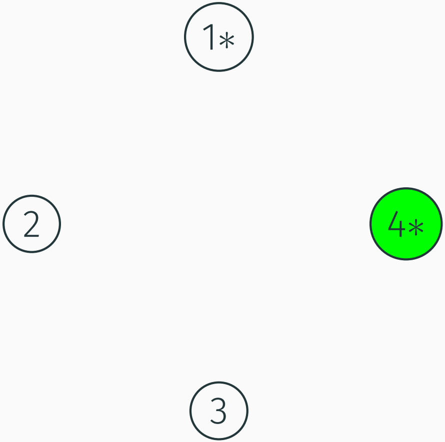

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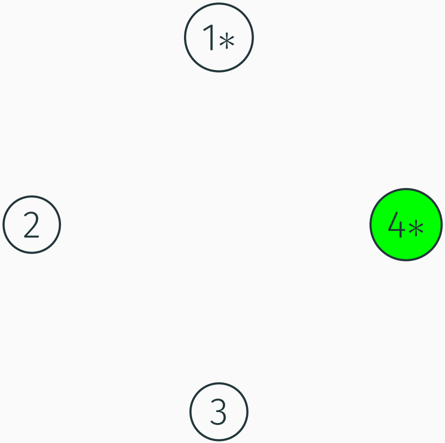

- Round 1
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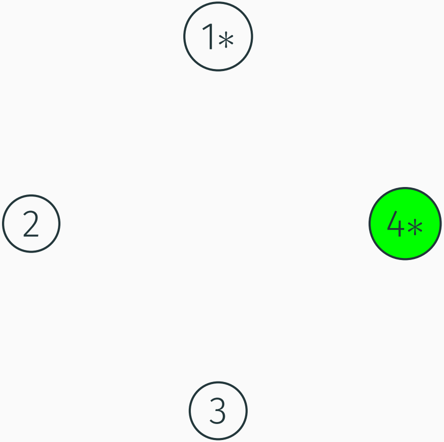
n = 4



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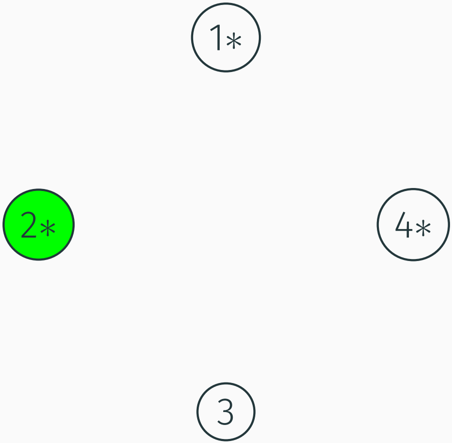

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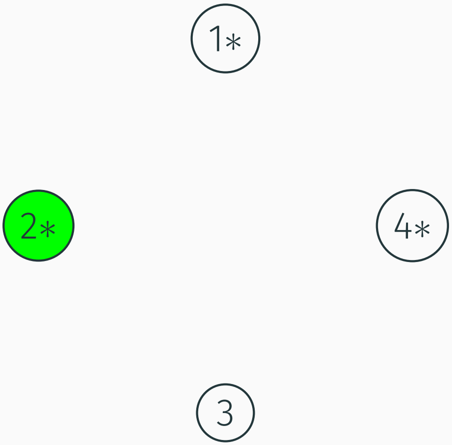

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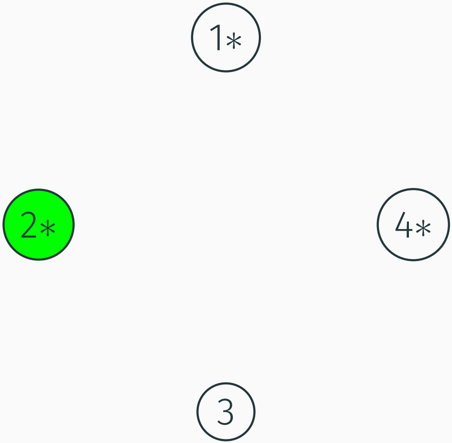

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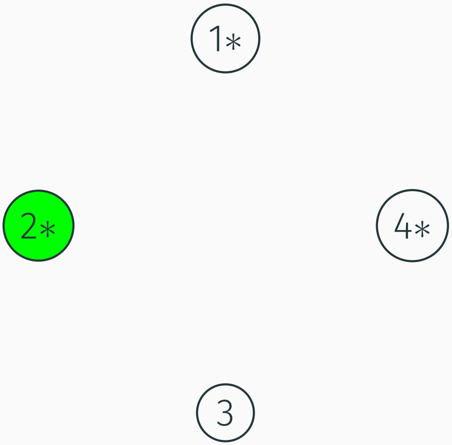

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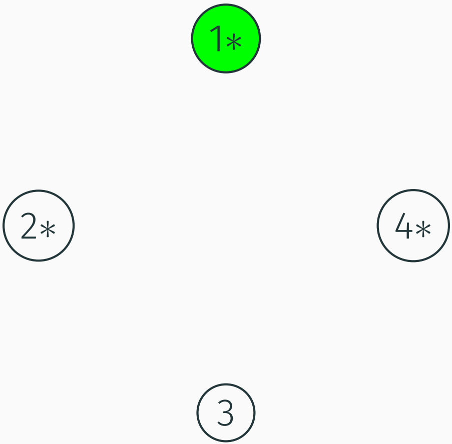

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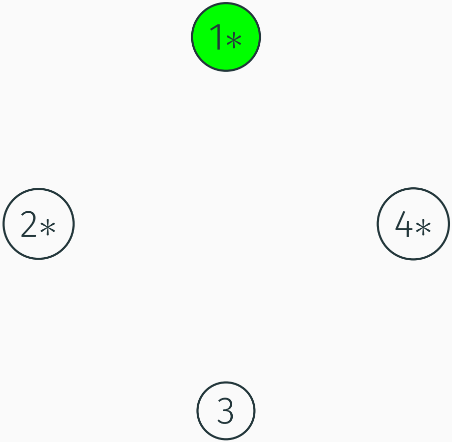

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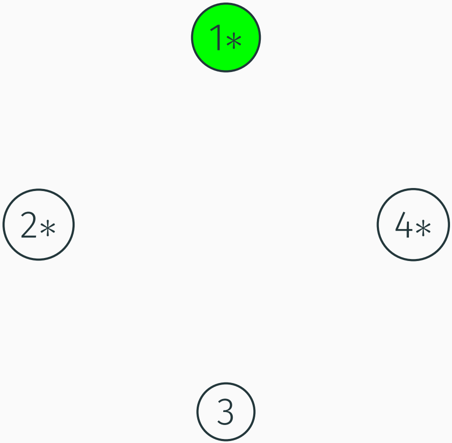

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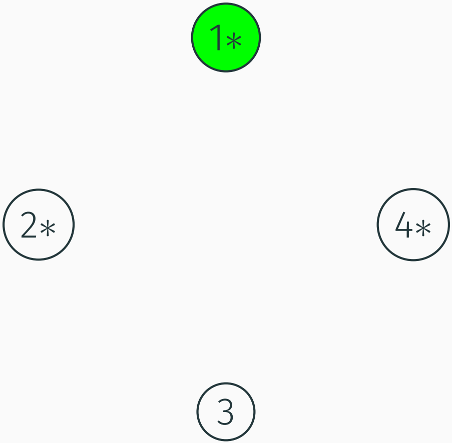

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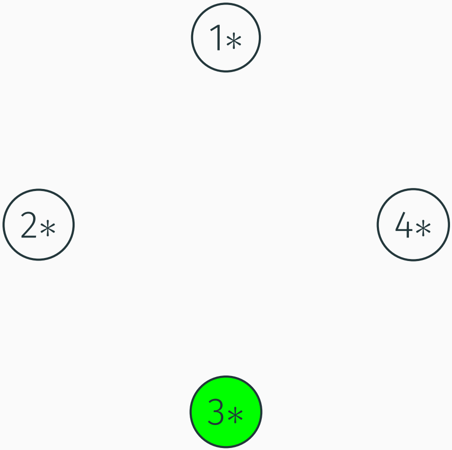

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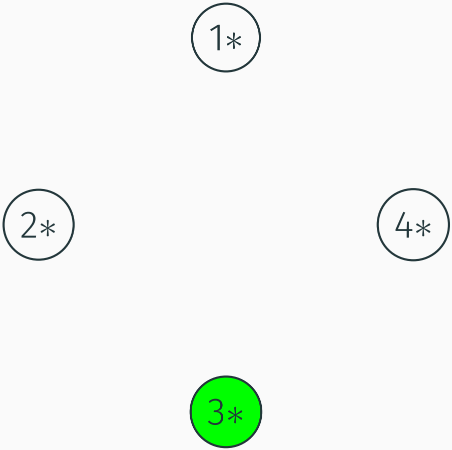

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General n

Nedev & Muthukrishnan, 2008:

$$f^*(n) = \begin{cases} n & \text{if } n \text{ is a power of 2} \\ n(1 - 1/p) & \text{if } p \text{ is the smallest odd divisor of } n \end{cases}.$$

The Magnus-Derek game in groups

Generalization

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- What is $f(G)$ for a general, finite group G?

Reducing to odd order groups

Proposition. Let G be a finite group and define

$$\Gamma = \langle \{g \in G : \mathrm{ord}(g) = 2^k \text{ for some } k\} \rangle.$$

Then $\Gamma \lhd G$ and

$$f(G) = |\Gamma| f(G/\Gamma).$$

Further, $|G/\Gamma|$ is odd.

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Then he can make the token visit at least $f(G/\Gamma)$ cosets by pretending to play in G/Γ; Derek does the same to make sure it visits at most $f(G/\Gamma)$ cosets.

Palindromes and main result

Definition. We say that a subset $P \subseteq G$ is *palindromic* if it satisfies

- $1 \in P$
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Theorem. If G is a group of odd order, then

$$f(G) = |G| - |P|$$

where $P \subsetneq G$ is a palindromic subset of maximal size.

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  - Derek wins if he can prevent this from happening.

Define $\tilde{f}(G) = |G| - \max_N |N|$ where the maximum is taken over all subsets N for which Derek can win.

Proof of main result

We want to show, that for a group G of odd order, $f(G) = |G| - |P|$ where P is palindromic in G of maximal size.

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- **Lemma 1**. $\tilde{f}(G) = f(G)$
- **Lemma 2**. Let $N \subseteq G$ be a set for which Derek can win, of maximal size. Then

$$xg, xg^{-1} \in N \Rightarrow x \in N. \quad (*)$$

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- **Lemma 3**. If G has odd order and $N \subseteq G$ satisfies the property $(*)$ from Lemma 2, then there exists an element $a \in G$ such that $N = aP$ where $P \subseteq G$ is palindromic.

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- **Lemma 4**. If G has odd order and $P \subsetneq G$ is palindromic of maximal size, then there exists an element $a \in G$ such that Derek can pick the set aP and win.

Proof of main result

Lemma 1. $\tilde{f}(G) = f(G)$

Proof. In the original game, Derek can pick a maximal set N for which he can win the open game without telling Magnus, and play as if playing the open game. Thus $f(G) \leq \tilde{f}(G)$.

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Now consider the original game from Magnus's point of view. Suppose, at the current step, that N is the set of elements the token hasn't visited. If $|N| > |G| - \tilde{f}(G)$, then Magnus can pretend that Derek has picked the set N, and play as in the open game to make the token reach some element of N. Eventually, the size of N will shrink to $|G| - \tilde{f}(G)$, which means that the token will have reached $\tilde{f}(G)$ elements. Thus $f(G) \geq \tilde{f}(G)$.

Proof of main result

Lemma 2. Let $N \subseteq G$ be a set for which Derek can win, of maximal size. Then

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Now suppose $x \notin N$. Then Magnus can move the token to x and choose g, forcing Derek to move it to xg or xg^{-1}, contradicting that both belong to N.

Definition. Let $a, b, c \in G$. We say that b is *between* the elements a and c if there exist $x, g \in G$ such that $a = xg^{-1}$, $b = x$ and $c = xg$.

Proof of main result

Definition. Let $a, b, c \in G$. We say that b is *between* the elements a and c if there exist $x, g \in G$ such that $a = xg^{-1}$, $b = x$ and $c = xg$.

Proposition. Suppose G has odd order and let $x, y \in G$. If $2m - 1$ is the order of $y^{-1}x$, then $B(x, y) := y(y^{-1}x)^m$ is the unique element between x and y.

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We can now state **Lemma 2** as follows: Let $N \subseteq G$ be a set for which Derek can win, of maximal size. Then

$$x, y \in N \Rightarrow B(x, y) \in N. \quad (*)$$

Proof of main result

Lemma 3. If G has odd order and $N \subseteq G$ satisfies the property $(*)$ from Lemma 2, then there exists an element $a \in G$ such that $N = aP$ where $P \subseteq G$ is palindromic.

Proof. We will use the following statement, without proof:

$$\text{If } a, ax \in N, \text{ then } ax^k \in N \text{ for all } k. \quad (**)$$

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Proof. We will use the following statement, without proof:

$$\text{If } a, ax \in N, \text{ then } ax^k \in N \text{ for all } k. \quad (**)$$

Now fix some $a \in N$ and take $x, y \in a^{-1}N$. We want to show that $xyx \in a^{-1}N$. Since $a, ay \in N$, by $(**)$ we have

$$ax(x^{-1}y^{-1}) = ay^{-1} \in N$$

and since $ax, ax(x^{-1}y^{-1}) \in N$, by $(**)$ again we have

$$axyx = ax(x^{-1}y^{-1})^{-1} \in N,$$

i.e. $xyx \in a^{-1}N$

Proof of main result

Lemma 4. If G has odd order and $P \subsetneq G$ is palindromic of maximal size, then there exists an element $a \in G$ such that Derek can pick the set aP and win.

Proof. Suppose the token starts at $g_0 \in G$. Pick a such that $g_0 \notin aP$. We want to show that $B(ax, ay) \in aP$ for any $x, y \in P$. Note that $B(ax, ay) = aB(x, y)$ and

$$B(x,y) = y(y^{-1}x)^m = x\,\underbrace{y^{-1}x \cdots xy^{-1}x}_{m-1 \text{ factors of } y^{-1}x}$$

is a palindrome in x, y. Thus $B(x, y) \in P$ as desired.

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Suppose that the token is currently at $g \in G \setminus aP$. Then $g = B(gh, gh^{-1})$ for any $h \in G$ and thus either $gh \notin aP$ or $gh^{-1} \notin aP$. Hence Derek can keep the token outside of aP as desired.

More on palindromes in groups

Goal

We say that a finite group G is *nilpotent* if for every divisor d of $|G|$, there exists a subgroup H of G such that $|H| = d$.

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This will imply, that for a nilpotent group G of odd order, $f(G) = |G|(1 - 1/p)$ where p is the smallest divisor of $|G|$, since we can just pick P as a subgroup of index p in G.

Reversibility

Definition. Let $G = \langle X \rangle$ be a group. For a word w written in the alphabet X, denote by $|w|$ the corresponding group element, and by $\overline{w}$ the word obtained by reversing the order of the letters in w. We sat that G is X-reversible if

$$|\overline{w_1}| = |\overline{w_2}| \iff |w_1| = |w_2|$$

for all words w_1, w_2 in X. If G is reversible, we write $\overline{g}$ for the unique element $|\overline{w}|$ where $|w| = g$.

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Proposition. (Fink & Thom, 2015) $H := \{|\overline{w}| \in G : |w| = 1\}$ is a normal subgroup of G.

Proposition. The subgroup H above is trivial if and only if G is X-reversible.

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Lemma. Let $G = \langle X \rangle$ be a group of odd order and suppose G is X-reversible. Let $N = \{g \in G : \overline{g} = g^{-1}\}$ and $P = \{g \in G : \overline{g} = g\}$ (i.e. P is the set of X-palindromes of G). Then every element of G can be written uniquely as pn where $p \in P$ and $n \in N$.

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Corollary. The number of elements of G which can be written as X-palindromes, divides $|G|$.

Proof. True if G is X-reversible, since N from the Lemma is a subgroup.

If G is not X-reversible, consider the group G/H where $H = \{|\overline{w}| \in G : |w| = 1\}$. If G/H is X/H-reversible, then the number of X/H-palindromes of G/H divides $|G|/|H|$. For an X/H-palindrome pH, we then obtain $|H|$ different X-palindromes $p|w\overline{w}| = |w|p|\overline{w}|$ of G (here, $|w\overline{w}| = |\overline{w}| \in H$, where $|w| = 1$). If not, repeat and take the quotient by $H_2 = \{|\overline{w}| \in G/H : |w| = 1\}$.

Conclusion

Recall that $f(G) = |G| - |P|$, where P is a palindromic subset of G, i.e. a set satisfying

- $1 \in P$
- $a, b \in P \Rightarrow aba \in P$.

Therefore, P is the set of all P-palindromes of $\langle P \rangle$, so $|P|$ divides $|\langle P \rangle|$ which divides $|G|$.

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Therefore, P is the set of all P-palindromes of $\langle P \rangle$, so $|P|$ divides $|\langle P \rangle|$ which divides $|G|$.

The largest possible size of a palindromic subset of G is thus $|G|/p$ where p is the smallest divisor. Since subgroups are palindromic, then if G is nilpotent, $f(G) = |G|(1 - 1/p)$.