The Foundations of Condensed Mathematics

Stage M2 mathématiques fondamentales
Université de Paris

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Sieves

- Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. A *sieve on X* is a subfunctor S of the functor $\mathrm{Hom}_{\mathcal{C}}(-, X) : \mathcal{C} \to \mathsf{Set}$.
- In other words, $S(Y) \subset \mathrm{Hom}_{\mathcal{C}}(Y, X)$, for all $f \in S(Y)$ and all $g : Z \to Y$, we have $f \circ g \in S(Z)$.

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- In other words, $S(Y) \subset \mathrm{Hom}_{\mathcal{C}}(Y,X)$, for all $f \in S(Y)$ and all $g : Z \to Y$, we have $f \circ g \in S(Z)$.
- Let $f : Y \to X$; the *pullback of S along f* is the sieve f^*S on Y:

$$f^*S(Z) = \{g : Z \to Y \mid f \circ g \in S(Z)\}$$

- Let $F = \{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms. The sieve S *generated by F* is defined by

$$S(Y) = \{f : Y \to X \mid f \text{ factors through some } f_i \in F\}.$$

The sheaf condition 1

- A *presheaf* on $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}.$$

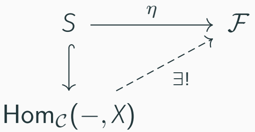
  For $f : U \to V$, the morphism $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U)$ is denoted by f^*.

- Let $\mathcal{F}$ be a presheaf and S a sieve on $X \in \mathcal{C}$. Then $\mathcal{F}$ *satisfies the sheaf condition with respect to* S if the map

$$\mathsf{Nat}(\mathsf{Hom}(-, X), \mathcal{F}) \to \mathsf{Nat}(S, \mathcal{F})$$

  induced by the inclusion $S \hookrightarrow \mathsf{Hom}(-, X)$ is a bijection.

- In other words, if every natural transformation $\eta : S \to \mathcal{F}$ has a unique extension to a natural transformation $\mathsf{Hom}(-, X) \to \mathcal{F}$:



The sheaf condition 2

Let $\mathcal{F}$ be a presheaf and S a sieve on $X \in \mathcal{C}$. Suppose S is generated by a family $F = \{f_i : X_i \to X\}_{i \in I}$ and that the appearing fibre products exist. Then the sheaf condition is equivalent to:

- The diagram

$$\mathcal{F}(X) \xrightarrow{e} \prod_{i \in I} \mathcal{F}(X_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(i,j) \in I \times I} \mathcal{F}(X_i \times_X X_j)$$

is an equaliser diagram. The maps are defined as follows:

$$e(x) = (f_i^*(x))_{i \in I} \qquad \text{for } x \in \mathcal{F}(X);$$

$$p_1(\mathbf{x})_{i,j} = \pi_{ij,1}^*(x_i) \text{ and } p_2(\mathbf{x})_{i,j} = \pi_{ij,2}^*(x_j) \qquad \text{for } \mathbf{x} = (x_i)_{i \in I} \in \prod_{i \in I} \mathcal{F}(X_i)$$

where $\pi_{ij,1} : X_i \times_X X_j \to X_i$ and $\pi_{ij,2} : X_i \times_X X_j \to X_j$ are the projections.

Sites – idea

- A site is a category equipped with a collection of *covering sieves* or *covering families* on each object, satisfying certain axioms.

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- The most important point is defining a notion of *sheaf* on the site.
  - These are the presheaves which satisfy the sheaf condition with respect to every covering sieve of each object.
- Covering sieves can be made to satisfy different axioms, giving different coverages, but the same sheaves.

Sites – coverage

Definition. Let $\mathcal{C}$ be a category. A *coverage* τ on $\mathcal{C}$ is given by specifying a set $\mathsf{Cov}_\tau(X)$ of *covering sieves* for each object X, satisfying

- If $S \in \mathsf{Cov}_\tau(X)$ and $f : Y \to X$, then there is a sieve $R \subset f^*S$ such that $R \in \mathsf{Cov}_\tau(Y)$.

Sites – Grothendieck topology

Definition. Let $\mathcal{C}$ be a category. A *Grothendieck topology* $\mathcal{T}$ on $\mathcal{C}$ is given by specifying a set $\mathrm{Cov}_{\mathcal{T}}(X)$ of *covering sieves* for each object X, satisfying

(1) (Identity) For all objects X, $\mathrm{Hom}(-, X) \in \mathrm{Cov}_{\mathcal{T}}(X)$.

(2) (Base change) If $S \in \mathrm{Cov}_{\mathcal{T}}(X)$ and $f : Y \to X$, then $f^*S \in \mathrm{Cov}_{\mathcal{T}}(Y)$.

(3) (Local character) If $S \in \mathrm{Cov}_{\mathcal{T}}(X)$ and R is a sieve on X such that

$$f^*R \in \mathrm{Cov}_{\mathcal{T}}(Y)$$

for all objects Y and all $f \in S(Y)$, then $f^*R \in \mathrm{Cov}_{\mathcal{T}}(Y)$, then $R \in \mathrm{Cov}_{\mathcal{T}}(X)$.

Sites – Grothendieck pretopology

Definition. A *Grothendieck pretopology* $\mathcal{P}$ on a category $\mathcal{C}$ is given, for each object X of $\mathcal{C}$, by a set $\mathrm{Cov}_{\mathcal{P}}(X)$ of families $\{X_i \to X\}_{i\in I}$ of morphisms, satisfying

(1) If $Y \to X$ is an isomorphism then $\{Y \to X\} \in \mathrm{Cov}_{\mathcal{P}}(X)$.

(2) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}_{\mathcal{P}}(U)$ and $\{Y_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}_{\mathcal{P}}(X_i)$ for all $i \in I$, then the family of compositions $\{Y_{ij} \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}_{\mathcal{P}}(X)$

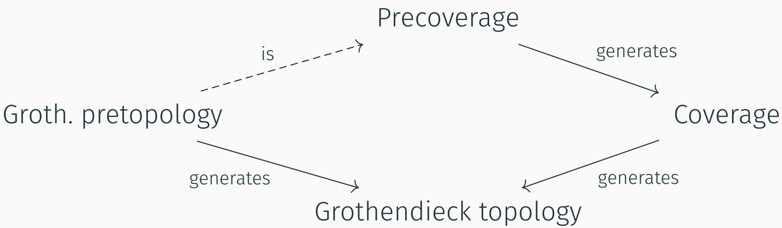
(3) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}_{\mathcal{P}}(X)$ and $Y \to X$ is a morphism of $\mathcal{C}$ then $X_i \times_X Y$ exists for all i and $\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}_{\mathcal{P}}(Y)$.

Sites – precoverage

Let $\mathcal{C}$ be a category. A *precoverage* π on $\mathcal{C}$ is given by specifying a set $\mathsf{Cov}_\pi(X)$ of *covering families* $\{X_i \to X\}_{i \in I}$ of morphisms with target X, satisfying the following condition

- If $\{f_i : X_i \to X\}_{i \in I} \in \mathsf{Cov}_\pi(X)$ and $g : Y \to X$ is any morphism, then there exists a $\{h_j : Y_j \to Y\}_{j \in J} \in \mathsf{Cov}_\pi(Y)$ such that each $g \circ h_j$ factors through an f_i.

Generation


Precoverage
is
generates
Groth. pretopology
Coverage
generates
generates
Grothendieck topology

Sheaves

- A *sheaf* on a site $\mathcal{C}$ is a presheaf satisfying the sheaf condition with respect to every covering sieve.
- If the coverage is generated by a precoverage for which the relevant fibre products exist (for example a Grothendieck pretopology), then it is enough to check that the covering families satisfy the second sheaf condition.

Sites of compact Hausdorff spaces

- A Grothendieck pretopology on the category of compact Hausdorff spaces (CHaus) is given by the finite, jointly surjective families.
- This remains a Grothendieck pretopology when restricted to the subcategory of profinite sets (Prof).
- The subcategory of extremally disconnected sets (ED) is not closed under fibre products. However, the finite, jointly surjective families form a precoverage on ED.

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- For all of the above sites, the precoverage consisting of the following two types of families
  1. $\{f_i : S_i \to S\}_{i \in I}$ such that I is finite and the induced $\coprod_{i \in I} S_i \to S$ is an isomorphism
  2. singleton families $\{p : S' \to S\}$ where $p : S' \to S$ is surjective.

  generates the same Grothendieck topology.

Condensed sets

- The categories of sheaves on the three sites are equivalent via restriction:

$$\mathsf{Sh}(\mathsf{CHaus}) \simeq \mathsf{Sh}(\mathsf{Prof}) \simeq \mathsf{Sh}(\mathsf{ED})$$

- A sheaf on one of the three sites is called a *condensed set*.

A simpler characterisation of condensed sets

Theorem. Let $\mathcal{C}$ be one of the sites CHaus or Prof. Then a presheaf T on $\mathcal{C}$ is a sheaf if and only if it satisfies the following two conditions.

(i) For any finite collection $(S_i)_{i \in I}$ of objects of $\mathcal{C}$, the natural map

$$T\left(\coprod_{i \in I} S_i\right) \to \prod_{i \in I} T(S_i)$$

is a bijection.

(ii) For any surjection $S' \to S$ of profinite sets, let p_1 and p_2 denote the two projections $S' \times_S S' \to S'$. Then the map

$$T(S) \to \{x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection.

An even simpler characterisation!

Theorem. A presheaf T on ED is a sheaf if and only if it satisfies the following condition

(i) For any finite collection $(S_i)_{i \in I}$ of objects of ED, the natural map

$$T\left(\coprod_{i \in I} S_i\right) \to \prod_{i \in I} T(S_i)$$

is a bijection.

Underlying set, associated condensed set

- Every condensed set T has an *underlying set* $T(*)$.
- For any topological space T, the presheaf $S \mapsto \underline{T}(S) = C(S, T)$ is a sheaf (condensed set).
- These form an adjoint pair when restricted to a certain subcategory of topological spaces (compactly generated).
- The inclusion of sheaves of a site in its presheaves has a left adjoint called *sheafification*.

Condensed abelian groups

Limits and colimits in condensed abelian groups

- A condensed abelian group is a presheaf of abelian groups on the site ED, such that finite disjoint unions are sent to the corresponding finite products (direct sums).
- Limits and colimits exist in CondAb and are computed objectwise on extremally disconnected sets: for S extremally disconnected and $I \to \mathrm{CondAb}$ $i \mapsto M_i$ a functor, we have

$$(\varinjlim_i M_i)(S) = \varinjlim_i M_i(S)$$

$$(\varprojlim_i M_i)(S) = \varprojlim_i M_i(S)$$

Compact generation and AB axioms

- The category CondAb is generated by compact projectives, more precisely by the condensed abelian groups of the form $\mathbb{Z}[\underline{S}]$ where S is extremally disconnected.
- CondAb satisfies all the same of Grothendiecks AB axioms as abelian groups (all limits and colimits exist, direct sums, products and filtered colimits are exact etc.)

Internal hom and tensor product

- We have a symmetric monoidal tensor product in condensed abelian groups: $M \otimes N$ is the sheafification of

$$S \mapsto M(S) \otimes N(S)$$

- It represents bilinear maps and has a right adjoint called *internal hom*, denoted $\underline{\mathrm{Hom}}$.
- This means that we have functorial isomorphisms

$$\mathrm{Hom}_{\mathrm{CondAb}}(N \otimes M, P) \simeq \mathrm{Hom}_{\mathrm{CondAb}}(N, \underline{\mathrm{Hom}}(M, P))$$

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  or more concretely, its S-valued points are

$$\begin{aligned} \underline{\mathrm{Hom}}(M, N)(S) &= \mathrm{Hom}_{\mathrm{CondSet}}(\underline{S}, \underline{\mathrm{Hom}}(M, N)) \\ &= \mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[\underline{S}], \underline{\mathrm{Hom}}(M, N)) \\ &= \mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[\underline{S}] \otimes M, N). \end{aligned}$$

Derived condensed abelian groups

- We can take the derived category of condensed abelian groups, $D(\mathrm{CondAb})$ as with any abelian category.
    - Objects are complexes of condensed abelian groups, quasi-isomorphisms (morphisms inducing isomorphisms in cohomology) between complexes become isomorphisms in the derived category.
- We have derived hom $R\,\mathrm{Hom}$, derived internal hom $R\underline{\mathrm{Hom}}$, and derived tensor product $\otimes^L$, satisfying the adjunction

$$R\,\mathrm{Hom}(M \otimes^L N, P) = R\,\mathrm{Hom}(M, R\underline{\mathrm{Hom}}(N, P))$$

Cohomology

Internal cohomology in the topos of condensed sets

- Let $S \in \mathsf{CHaus}$. Condensed cohomology of S: higher derived functors of

$$\Gamma(S, -) : \mathsf{CondAb} \to \mathsf{Ab},$$

  denoted $H^i_{\mathsf{cond}}(S, M)$ for $M \in \mathsf{CondAb}$.

- We have

$$\Gamma(S, -) = \mathsf{Hom}_{\mathsf{CondAb}}(\mathbb{Z}[\underline{S}], -)$$

  so condensed cohomology can be extended to condensed sets T; $H^i_{\mathsf{cond}}(T, M)$ is the cohomology of $R\,\mathsf{Hom}(\mathbb{Z}[T], M)$.

- For $S \in \mathsf{CHaus}$, take a hypercover $S_\bullet \to S$ of extremally disconnected sets and compute cohomology of the complex

$$0 \to \Gamma(S_0, M) \to \Gamma(S_1, M) \to \Gamma(S_2, M) \to \cdots$$

Sheaf cohomology

- Let $\mathcal{F}$ be a sheaf of abelian groups on the compact Hausdorff space S in the classical sense. There is a classical notion of cohomology on S with respect to this sheaf:
- **Sheaf cohomology.** $H^i_{\text{sheaf}}(S, \mathcal{F})$: take an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and compute the cohomology of the complex

$$0 \to \Gamma(S, \mathcal{I}^0) \to \Gamma(S, \mathcal{I}^1) \to \Gamma(S, \mathcal{I}^2) \to \cdots$$

  (the higher right derived functors of the global sections functor $\Gamma(S, -)$).

Cohomology coefficients in a discrete group

Theorem. Let S be a compact Hausdorff space and M a discrete abelian group. There are natural isomorphisms

$$H^i_{\mathsf{sheaf}}(S, M) \cong H^i_{\mathsf{cond}}(S, \underline{M})$$

where on the left, M is regarded as the sheafification of the constant presheaf $U \mapsto M$.

Cohomology coefficients in $\mathbb{R}$

Theorem. Let S be a compact Hausdorff space. Then

$$H^i_{\mathrm{cond}}(S, \underline{\mathbb{R}}) = \begin{cases} C(S, \mathbb{R}) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}.$$

Cohomology coefficients in $\mathbb{R}$

Theorem. Let S be a compact Hausdorff space. Then

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This is a corollary of

Theorem. Let S be a compact Hausdorff space. For any simplicial hypercover $S_\bullet \to S$ by profinite sets S_i, the complex of Banach spaces

$$0 \to C(S, \mathbb{R}) \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \cdots$$

satisfies the following “quantitative” version of exactness: if $f \in C(S_i, \mathbb{R})$ satisfies $df = 0$, then for any $\varepsilon > 0$ there exists a $g \in C(S_{i-1}, \mathbb{R})$ such that $dg = f$ and $\|g\| \leq (i + 2 + \varepsilon)\,\|f\|$.

Locally compact abelian groups

Topological vs. condensed

Definition. A topological space X is said to be *compactly generated* if continuous maps $X \to Y$ are precisely those making the composite $S \to X \to Y$ continuous for every compact Hausdorff space S mapping continuously to X.

Theorem.

(i) The functor $X \mapsto \underline{X}$ from (κ-small) topological spaces to condensed sets is faithful, and fully faithful when restricted to the subcategory of compactly generated topological spaces.

(ii) The functor $X \mapsto \underline{X}$ admits a left adjoint $T \to T(*)$ where the underlying set $T(*)$ is equipped with the final topology for the collection of all maps $S \to T(*)$ where S is compact Hausdorff, that come from a map of condensed sets $\underline{S} \to T$ (here we regard condensed sets as sheaves on the site CHaus).

Locally compact abelian groups

Structure theorem for locally compact abelian groups. Let A be a locally compact abelian group. There exists an integer n and a locally compact abelian group A' admitting a compact open subgroup such that

$$A \simeq \mathbb{R}^n \times A'.$$

Pontrjagin duality. Let $\mathbb{T}$ denote the circle group $\mathbb{R}/\mathbb{Z}$. The functor $\mathbb{D}$, which takes a locally compact abelian group to the abelian group $\mathrm{Hom}(A, \mathbb{T})$ equipped with the compact-open topology, takes values in LCA and induces a contravariant autoequivalence of LCA. The map $A \to \mathbb{D}(\mathbb{D}(A))$ is an isomorphism. Moreover, $\mathbb{D}$ restricts to a duality from compact abelian groups to discrete abelian groups.

Compact-open topology and condensed internal hom

Proposition. Let A and B be Hausdorff topological groups with A compactly generated. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \simeq \underline{\mathrm{Hom}(A, B)}$$

where $\mathrm{Hom}(A, B)$ is equipped with the compact-open topology.

In particular, this holds for locally compact abelian groups A and B.

Derived homs between locally compact abelian groups

- There is a notion of bounded derived category of the quasi-abelian category of locally compact abelian groups. The notion of $R\operatorname{Hom}$ in this category can be shown to agree with the condensed $R\operatorname{Hom}$. Thanks to the structure theorem, the calculation can be reduced to the following theorem.

The theorem

Theorem. Consider the condensed abelian group associated to a compact abelian group consisting of a product of circles, $A = \underline{\prod_I \mathbb{T}} = \underline{\prod_I \mathbb{R}/\mathbb{Z}}$ where I is any set. We have the following:

For any discrete condensed abelian group M (i.e. $M = \underline{M'}$ where M' is a discrete abelian group),

$$R\underline{\mathrm{Hom}}(A, M) = \bigoplus_I M[-1]$$

where the isomorphism

$$\bigoplus_I M[-1] \to \underline{R\mathrm{Hom}(A, M)}$$

is induced by the maps

$$M[-1] = R\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}[1], M) \to R\underline{\mathrm{Hom}}(\underline{\mathbb{R}/\mathbb{Z}}, M) \to R\underline{\mathrm{Hom}}(A, M),$$

where the last map is induced from the projection $p_i : \prod \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to the i-th factor, $i \in I$.

Further, $R\underline{\mathrm{Hom}}(A, \mathbb{R}) = 0$.

Thanks for listening!