Cranks and t-cores

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Abstract

We explain and clarify the work of Garvan et al. [GKS90] on Dyson's crank of integer partitions [Dys44], proving Ramanujan's congruences.

1 Introduction

Recall that a *partition* of a positive integer n is a non-increasing finite sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $n = \lambda_1 + \cdots + \lambda_k$. For $i = 1, \dots, k$, λ_i is called a *part* of the partition. The number of partitions of n is denoted $p(n)$. A partition may be represented by a so-called Young diagram, i.e. an array of cells such that the i-th line has λ_i cells for $i = 1, \dots, k$ (see Figure 1). It is clear that there is a one-to-one

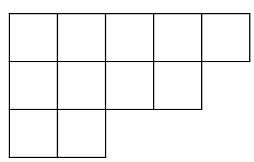

Figure 1: Young diagram of the partition $\lambda = (5, 4, 2)$.

correspondence between partitions and Young-diagrams. This justifies the definition of the conjugate partition:

Let λ be a partition of n. Then we define the *conjugate* partition λ' such that it has λ_1 parts and λ'_i is the number of cells in the i-th column of the Young diagram of λ. In other words, λ' is the Young diagram obtained by transposing (i.e. reflecting about the main diagonal) the Young diagram λ.

Theorem 1.1. *(Ramanujan's congruences). For $n \in \mathbb{N}$,*

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

Theorem 1.1 was first proved by Ramanujan in [Ram21], and has since been proved in many different ways, see for example [HW08].

In [Dys44], Dyson noted that all existing proofs of Ramanujan's congruences relied on generating function identities which gave no idea of how to split the partitions of $5n+4$ (resp. $7n+5$ and $11n+6$) into 5 (resp. 7 and 11) equinumerous classes. He proceeds to define the *rank* of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ as $\operatorname{rank}(\lambda) = \lambda_1 - k$. By definition, $\operatorname{rank}(\lambda) = -\operatorname{rank}(\lambda')$. Let P denote the set of all partitions, $N(m,n)$ the number of partitions of n of rank m and $N(m,t,n)$ the number of partitions of n of rank congruent to m modulo t. Since the map $P \to P$, $\lambda \mapsto \lambda'$ is bijective (it is its own inverse), we obtain that $N(m,n) = N(-m,n)$ and $N(m,t,n) = N(t-m,t,n)$. Dyson conjectured that the rank would split the partitions of $5n+4$ and $7n+5$ into 5 and 7 equinumerous groups respectively. More precisely, that

$$N(m,5,5n+4) = \frac{p(5n+4)}{5}$$

for $m = 0,1,2,3,4$ and

$$N(m,7,7n+5) = \frac{p(7n+5)}{7}$$

for $m = 0,1,2,3,4,5,6$. He noted that this strategy wouldn't work for partitions of $11n+6$, but nevertheless suggested that something similar might work. He defined the term *crank* as a statistic $\operatorname{crank}(\lambda)$ of a partition λ, such that if $M(m,t,n)$ denotes the number of partitions λ of n such that $\operatorname{crank}(\lambda) \equiv m \pmod{t}$, the following should hold:

$$M(m,t,n) = M(t-m,t,n) \tag{1}$$

and

$$M(m,t,tn+r) = \frac{p(tn+r)}{t} \tag{2}$$

for $m \in \{0,\ldots,t-1\}$ where $(t,r) \in \{(5,4),(7,5),(11,6)\}$. Atkin and Swinnerton-Dyer proved Dyson's conjecture about the rank for partitions of $5n+4$ and $7n+5$ in [AS54]. Cranks were then found in [AG88; Gar88]. Later, Garvan, Kim and Stanton [GKS90] gave a single strategy to find cranks for $5n+4$, $7n+5$ and $11n+6$, along with explicit bijections between the crank classes. The aim of this thesis is to explain and clarify their work.

In Section 2, the prerequisite results and definitions from partition theory are given. In Section 3, we define two bijections relating partitions and vectors in $\mathbb{Z}^t$. These are crucial ingredients in the definition of the cranks given in Section 4.

2 Background: hooks and t-cores

Definition 2.1. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ its conjugate.

1. The (i, j)-*cell of* λ is the cell in row i and column j of the Young-diagram of λ.

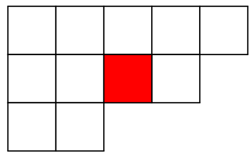

Figure 2: (2,3)-cell of the partition $\lambda = (5, 4, 2)$.

2. The (i, j)-*hook of* λ is the subset consisting of the (i, r)- and (s, j)-cells of λ with $r \geq j$ and $s \geq i$. The (i, j)-hook of λ is denoted H^{λ}_{ij}.

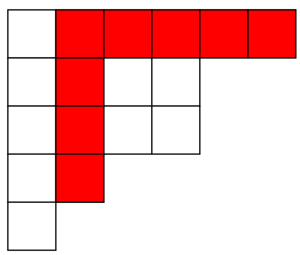

Figure 3: (1,2)-hook of the partition $\lambda = (6, 4, 4, 2, 1)$.

3. The number $h^{\lambda}_{ij} = \lambda_i - i + \lambda'_j - j + 1$ of cells in H^{λ}_{ij} is called the *length* of H^{λ}_{ij}.
4. The (i, j)-cell is said to be on the *rim* it is the last in a north-west to south-east diagonal (i.e. if the $(i + 1, j + 1)$-cell does not exist).
5. The set of (r, s)-cells on the rim of λ such that $i \leq r \leq \lambda_i$ and $j \leq s \leq \lambda'_j$ is denoted R^{λ}_{ij} and called *the associated part of the rim* or the *rim-(i, j)-hook* of λ.

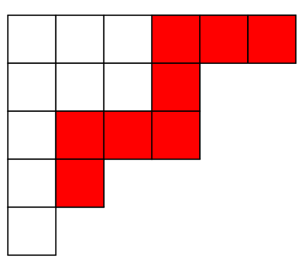

Figure 4: Rim-(1,2)-hook of the partition $\lambda = (6, 4, 4, 2, 1)$.

Remark 2.2. By picking a hook of λ and removing the associated part of the rim, it is easy to see that one obtains a new Young-diagram. It is also clear that the number of cells in R_{ij}^{λ} is the same as in H_{ij}^{λ}.

Definition 2.3. Let λ be a partition. The (i,j)-hook of λ is called a *t-hook* if $h_{ij}^{\lambda} = t$ and the associated part of the rim is called a *rim-t-hook*. The partition λ is said to be a t-core if it has no hooks of length a multiple of t, or equivalently no rim hooks of length a multiple of t.

Theorem 2.4. *([JK81, Theorem 2.7.16]). Pick a number t and a partition λ. By subsequent removal of rim-t-hooks from λ, one eventually obtains a t-core partition $\tilde{\lambda}$, independent of the sequence of removals. This unique partition $\tilde{\lambda}$ is called the t-core of λ.* □

Remark 2.5. Since hook lengths are preserved by conjugation, Theorem 2.4 shows that the t-core of the conjugate of a partition is the conjugate of the t-core of the original partition.

We illustrate Theorem 2.4 in Figure 5 where the 3-core of $\lambda = (6,4,4,2,1)$ is found: $\tilde{\lambda} = (3,1,1)$.

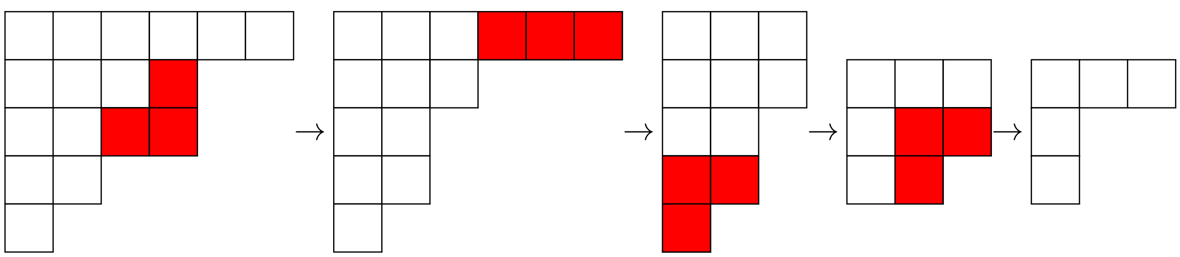

Figure 5: Sequence of rim-3-hook removals of the partition $\lambda = (6,4,4,2,1)$.

Definition 2.6. The *Durfee square* of a partition λ is largest square fitting inside the Young-diagram of λ with one vertex at $(1,1)$. The *size* of the Durfee square is the greatest integer s such that λ has at least s parts of value at least s.

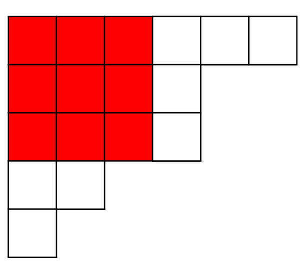

Figure 6: Durfee square of size 3 in the partition $\lambda = (6,4,4,2,1)$.

3 Two bijections

If λ is a partition, we denote the number which λ partitions by $|\lambda|$. We note by P the set of all partitions and $P_{t-\text{core}}$ the subset of all partitions which are t-cores. Recall that

$$p(n) := \#\{\lambda \in P : |\lambda| = n\}$$

and

$$a_t(n) := \#\{\lambda \in P_{t-\text{core}} : |\lambda| = n\}.$$

Finally, we shall use the notation

$$(a;q)_k := \prod_{i=0}^{k-1}(1 - aq^i)$$

(allowing $k = \infty$). Note that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Theorem 3.1. *([GKS90, Bijections 1 and 2]). There is a bijection*

$$\phi_1 : P \to P_{t-core} \times P \times \cdots \times P, \quad \lambda \mapsto (\tilde{\lambda}, \lambda_0, \ldots, \lambda_{t-1}),$$

such that

$$|\lambda| = |\tilde{\lambda}| + t\sum_{i=0}^{t-1} |\lambda_i|$$

(it should be noted that the λ_i are not the parts of λ, rather they are themselves partitions of smaller numbers). The generating function identity given by ϕ_1 is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^t;q^t)_\infty^t} \sum_{n=0}^{\infty} a_t(n)q^n.$$

There is another bijection

$$\phi_2 : P_{t-core} \to \{\vec{n} = (n_0, \ldots, n_{t-1}) \in \mathbb{Z}^t : n_0 + \cdots + n_{t-1} = 0\}, \quad \tilde{\lambda} \mapsto \vec{n},$$

where

$$|\tilde{\lambda}| = t\,\|\vec{n}\|^2/2 + \vec{b}\vec{n}, \quad \vec{b} = (0, 1, \ldots, t-1).$$

The generating function identity given by ϕ_2 is

$$\sum_{n=0}^{\infty} a_t(n)q^n = \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}}.$$

Proof. We begin by ϕ_2. We let $\tilde{\lambda}$ be a t-core and we define $\phi_2(\tilde{\lambda}) = \vec{n} = (n_0, \ldots, n_{t-1})$ as follows. We label the (i,j)-cell of $\tilde{\lambda}$ with $j - i \bmod t$. Then we add an infinite column to the left of $\tilde{\lambda}$ (the 0-th column) and label its cells in the same way. This diagram is called *the extended t-residue diagram*. A cell of the extended t-residue diagram is called *exposed* if it is at the (right) end of a row. For $r \in \mathbb{Z}$ we define *region* r of the extended t-residue diagram to be the set of cells such that

$$r(t-1) \leq j - i < tr$$

For $i = 0, \ldots t-1$ we define n_i as the maximum number of a region containing an exposed cell labelled i. This number is well defined since column 0 contains infinitely many exposed cells. As an example, consider $\tilde{\lambda} = (4, 2)$. Then $n_0 = 2$, $n_1 = -1$ and $n_2 = -1$ (see Figure 7, where region -1 of the extended 3-residue diagram is coloured white, region 0 in red, region 1 in green and region 2 in blue).

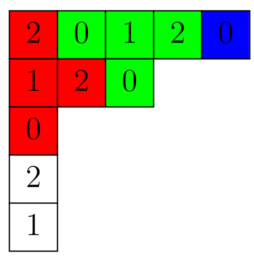

Figure 7: Extended 3-residue diagram of the 3-core $\tilde{\lambda} = (4, 2)$.

Note that by going down the rim (i.e. to the "south-west" or "down-left"), in each step one is either reducing j by 1 (taking a step to the left) or increasing i by 1 (taking a step down), in either case one is reducing $j - i$ by 1. Therefore, for all i, if region r has a cell labelled i on the rim, then all regions $< r$ have a cell labelled i on the rim.

Now let's prove that if i is exposed in region r, then the cell labelled i on the rim in region $r-1$ is exposed. First note that if the cell labelled i in region $r-1$ is in column 0, then it is exposed (all cells on the rim in column 0 are exposed). Now suppose our cell is not in column 0, i.e. is in the original partition. If it is not exposed, then there is a cell to the right of it, labelled $i+1 \bmod t$, which we call cell x. Since the cell labelled i is on the rim, there is no cell just below cell x. But then the part of the rim going from cell x up ("north-east") to the cell in region r labelled i is a rim-t-hook, contradicting the fact that $\tilde{\lambda}$ is a t-core. For later use, we note that if $\tilde{\lambda}$ is not a t-core, then for some r there is an exposed cell labelled i in region r and $s < r$ such that i is not exposed in region s. Indeed, take a rim-t-hook, and let i be the label on its north-eastermost cell, which we suppose to be in region r. Then by the argument just above, its south-westernmost cell (supposed to be in region $s < r$) will be labelled $i+1$. The cell labelled i on the rim in region s will then be to the left of the cell labelled $i+1$, and thus not exposed (the other option is that it is just below the cell labelled $i+1$, but that is not possible since $i+1$ is the foot (south-westernmost cell) of a rim hook).

We have just shown that there is a cell labelled i in every region numbered $\leq n_i$. Therefore the number of exposed cells in regions with positive number is the sum of the positive n_i's. The number of exposed cells in positive regions is also the number of rows which intersect a positive region. For a row to intersect a positive region, its length must be at least its number. Therefore, the number of exposed cells in positive regions is exactly the size of the Durfee square of $\tilde{\lambda}$. To show that $n_0 + \cdots + n_{t-1} = 0$, it suffices to show that if $\tilde{\lambda}'$ is the conjugate of $\tilde{\lambda}$, then

$$\phi_2(\tilde{\lambda}') = (-n_{t-1}, -n_{t-2}, \ldots, -n_0), \tag{3}$$

since the Durfee square is an invariant by conjugation, and that along with Equation (3) means that the negative sum of the negative n_i's is equal to the sum of the positive n_i's. Indeed, Equation (3) holds for $\tilde{\lambda}' = (2, 2, 1, 1)$, see Figure 8 (region -2 is coloured yellow).

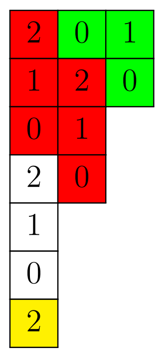

Figure 8: Extended 3-residue diagram of $\tilde{\lambda}' = (2, 2, 1, 1)$.

In general, consider an exposed cell in region r of $\tilde{\lambda}$ labelled i, such that there is a cell labelled $i+1$ above it. Suppose its coordinates are (k, m). Since

$$t(r-1) \leq k - 1 - m < tr,$$

we have

$$-tr \leq m - k < t(1-r),$$

and thus the corresponding cell in $\tilde{\lambda}'$ will be in region $1-r$, it will be on the rim, labelled $t-i-1$ and not exposed (since there it is not the last cell in its column in $\tilde{\lambda}$, it is not the last cell in its row in $\tilde{\lambda}'$). Conversely, if a cell on the rim of $\tilde{\lambda}'$ is in region $1-r$, labelled $t-i-1$ and not exposed, then the cell to its right corresponds to an exposed cell labelled i in region r of $\tilde{\lambda}$. We conclude that equation (3) holds.

To show that ϕ_2 is bijective, we give its inverse. Let $\vec{n} = (n_0, \ldots, n_{t-1})$ be given such that $n_0 + \cdots + n_{t-1} = 0$. Consider the lattice of all cells (i, j) with $i, j \in \mathbb{Z}$, where the (i, j)-cell is labelled $j - i \bmod t$. Define the regions as before. Then in each row, for $i = 0, \ldots, t-1$ there is exactly one cell labelled i in each region. Now for each integer m starting from the largest value of the n_i's down to the smallest, order the indices of the n_i's such that $n_i \geq m$ in decreasing order. Define a_k to be the k-th term of the sequence defined in this way and let $m(a_k)$ be the value of m when a_k was defined. Then define λ_i to be the column number of the cell labelled a_k in region $m(a_k)$ in row k. Then the positive λ_k's are the parts of $\tilde{\lambda} := \phi_2^{-1}(\vec{n})$. To illustrate this construction, consider Figure 9, where we have bolded the label of the exposed cell in each row of the resulting partition for $t = 3$ and $\vec{n} = (-1, 2, -1)$ The resulting

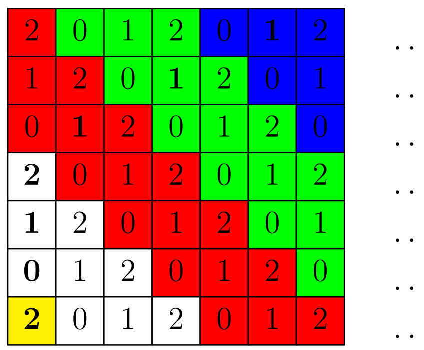

Figure 9: Lattice with regions -2 to 2 colored.

partition in the example of Figure 9 is $\phi_2^{-1}(\vec{n}) = (5, 3, 1)$. The sequence (a_k) in this case is $1, 1, 1, 2, 1, 0, 2, 1, 0, \ldots$.

We noted before that a partition is in fact a t-core if and only if the condition that the existence of a cell labelled i in region r implies the existence of a cell labelled i in region $r-1$ holds. Therefore, by construction, if ϕ_2^{-1} is well defined, then the resulting partition is a t-core. For it to be well defined, we need to show that the construction of the sequence a_k makes us end up in column number 0, i.e. at the point where $a_k = t-1, a_{k-1} = t-2, \ldots, a_{k+t-1} = 0$, then $\lambda_k = 0$. Let $r_{\min} = \min\{n_0, \ldots, n_{t-1}\}$. Then for each i, there are $n_i - r_{\min} + 1$ rows with a cell labelled i at the end, up to and including the first occurence of $a_k = t-1, a_{k-1} = t-2, \ldots, a_{k+t-1} = 0$ in the sequence. Therefore, there are in total

$$\sum_{i=0}^{t-1}(n_i - r_{\min} + 1) = -t(r_{\min} - 1)$$

rows (if we stop counting after the first occurence of $a_k = t-1, a_{k-1} = t-2, \ldots, a_{k+t-1} = 0$). Now, for all $i \in \{0, \ldots, t-1\}$ the cells labelled i in region $r_{\min}$ and rows $-t(r_{\min} - 1) - i$ are in the same column j such that

$$t(r_{\min} - 1) \leq j + t(r_{\min} - 1) + i < tr_{\min}$$

Plugging in $i = 0$, the former inequality gives $0 \leq j$ and plugging in $i = t-1$, the latter gives $j < 1$. We conclude that $j = 0$, and thus that ϕ_2^{-1} is well defined.

Now we prove that if $\phi_2(\tilde{\lambda}) = \vec{n}$, then $|\tilde{\lambda}| = t\,\|\vec{n}\|^2/2 + \vec{b} \cdot \vec{n}$. First we show that the number of cells strictly to the right of the main diagonal of the Young diagram of $\tilde{\lambda}$ is

$$\sum_{n_i>0}\left(in_i+t\binom{n_i}{2}\right).$$

Indeed, let a postive n_i be given. Then there are n_i rows which have an exposed cell labelled i to the right of the main diagonal. In the first such row, to the right of the main diagonal, there are n_i-1 blocks of t cells labelled $1,2,\ldots,t-1,0$ (the first block in region 1, the second in region 2 etc. up to region n_i-1), followed by the i cells labelled: $1,2,\ldots,i$. In general, in the m-th such row, to the right of the main diagonal there are n_i-m blocks of t cells labelled $1,2,\ldots,t-1,0$, followed by the i cells labelled: $1,2,\ldots,i$. Therefore the number of cells to the right of the main diagonal in the rows which end with an exposed cell labelled i is

$$\sum_{m=1}^{n_i}(t(n_i-m)+i)=in_i+\sum_{m=1}^{n_i}t(n_i-m)=in_i+t\binom{n_i}{2},$$

yielding the desired result. By applying the same argument on $\tilde{\lambda}'$ one obtains that the number of cells to the left of the main diagonal of the Young diagram of $\tilde{\lambda}$ is

$$-\sum_{n_j<0}\left((t-1-j)n_j-t\binom{-n_j}{2}\right).$$

We have already seen that the number of cells on the main diagonal is

$$\sum_{n_i>0}n_i.$$

Therefore, since the negative sum of the negative n_j's is equal to the sum of the postitve n_i's,

$$\begin{aligned}
|\tilde{\lambda}| &= \sum_{n_i>0}\left(in_i+t\binom{n_i}{2}\right)-\sum_{n_j<0}\left((t-1-j)n_j-t\binom{-n_j}{2}\right)+\sum_{n_i>0}n_i\\
&= \sum_{n_i}in_i+-(t-1)\sum_{n_j<0}n_j+\sum_{n_i>0}n_i+\frac{t}{2}\sum_{n_i>0}n_i(n_i-1)+\frac{t}{2}\sum_{n_j<0}n_j(n_i+1)\\
&= \vec{b}\cdot\vec{n}+\frac{t}{2}\sum_{n_i>0}n_i+\frac{t}{2}\sum_{n_i>0}n_i(n_i-1)-\frac{t}{2}\sum_{n_j<0}n_j+\frac{t}{2}\sum_{n_j<0}n_j(n_j+1)\\
&= \vec{b}\cdot\vec{n}+\frac{t}{2}\sum_{n_i}n_i^2=\vec{b}\cdot\vec{n}+\frac{t}{2}\left\|\vec{n}\right\|^2.
\end{aligned}$$

Now we give ϕ_1. Let λ be a partition, and $w_0,\ldots,w_{t-1}$ be t biinfinite words in the letters N and E, defined such that for $j\in\mathbb{Z}$, the j-th letter of w_i is E if the diagonal of i's in region j intersects the extended t-residue diagram of λ in an exposed cell, and otherwise it is N. Note that if the k-th letter of w_i is N and there is an E in position $j > k$, the diagonal of i's in region k intersects the rim of λ in a cell labelled i which is not exposed. In that case we call this cell the *cell corresponding to* N *in position* k. The *cell corresponding to* E *in position* j is of course the exposed cell in region j labelled i.

Now λ is a t-core if and only if each w_i is an infinite sequence of E's followed by an infinite sequence of N's. In that case, define $\tilde{\lambda} = \lambda$ and $\lambda_i = \varnothing$ for $i = 0, \ldots, t-1$. If λ is not a t-core, pick a w_i. Suppose the rightmost E is in position j and that the rightmost N to the left of the rightmost E is in position $k < j$. The cell corresponding to N in position k is on the rim of λ but not exposed. Therefore there is a cell to its right labelled $i+1$ which is the last in its column. The rim hook going from that cell to the cell corresponding to E in position $k+1$ is of length t. Remove it, and change w_i accordingly (the result is switching the N in position k and the E in position $k+1$). Repeat until the block of E's from position $k+1$ to j has been shifted one step to the left. Place a part of size $j-k$ in λ_i. Do this until not possible anymore; by Theorem 2.4 we end up with the t-core $\tilde{\lambda}$ of λ. Every time a new part is placed in λ_i in this process, $t(j-k)$ blocks are removed from λ, proving the identity

$$|\lambda| = |\tilde{\lambda}| + t\sum_{i=0}^{t-1} |\lambda_i|.$$

We illustrate the process on $\lambda = (6, 4, 4, 2, 1)$. Its extended 3-residue diagram is shown in Figure 10.

2	0	1	2	0	1	2
1	2	0	1	2		
0	1	2	0	1		
2	0	1				
1	2					
0						
2						
1						

Figure 10: Extended 3-residue diagram of $\lambda' = (6, 4, 4, 2, 1)$.

We show positions -2 to 2 of the words w_0, w_1, w_2:

$$w_0 = \ldots EENNN \ldots \qquad w_1 = \ldots ENEEN \ldots \qquad w_2 = \ldots EENEE \ldots$$

In all the words, the letters in positions ≥ 3 are N and those in positions ≤ -3 are E. No shifting occurs in w_0; therefore $\lambda_0 = \varnothing$. The shifting of letters in w_1 corresponds to removing the red rim-3-hook followed by the green one in Figure 11 (note that the green one doesn't become a rim hook until the red one is removed). The cell with the bold label corresponds to the N in position -1 and the cells with the italic labels correspond to the E's in position 0 and 1.

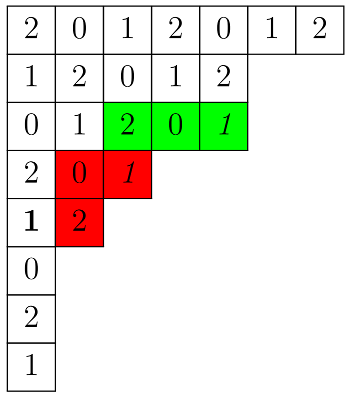

Figure 11: Removal of rim hooks corresponding to w_1 for $\lambda' = (6, 4, 4, 2, 1)$.

A part of size $1 - (-1) = 2$ is then added to λ_1. As a result w_1 is of the desired final form and thus $\lambda_1 = (2)$. Figure 12 shows the removal of rim hooks corresponding to w_2 in the same way.

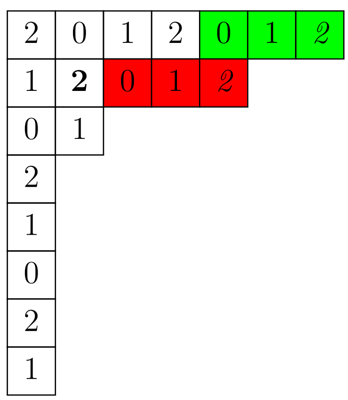

Figure 12: Removal of rim hooks corresponding to w_2 for $\lambda' = (6, 4, 4, 2, 1)$.

We conclude that $\lambda_2 = (2)$ and

$$\phi_1(\lambda) = ((3, 1, 1), \varnothing, (2), (2)).$$

For a proof that ϕ_1 is bijective, see [JK81, Theorem 2.7.17]. □

The two generating function identities in Theorem 3.1 yield Equation (4)

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q^t; q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{1}=0 \\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}} \tag{4}$$

4 Cranks

Since each term of the left factor of

$$\frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{1}=0\\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2+\vec{b}\cdot\vec{n}}$$

has exponent divisible by t, when it is expanded, the residue mod t of each exponent will be determined by the right factor. Furthermore, since $\vec{n}\cdot\vec{1}=0$, the number of odd n_i's is even, and thus the number of odd n_i^2's is even, so $\|\vec{n}\|^2$ is even and hence

$$\frac{t}{2}\|\vec{n}\|^2 \equiv 0 \bmod t$$

Therefore, Equation 4 yields

$$\sum_{n=0}^{\infty} p(tn+r)q^{tn+r} = \frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{\vec{n}\cdot\vec{b}\equiv r \bmod t\\ \vec{n}\cdot\vec{1}=0\\ \vec{n}\in\mathbb{Z}^t}} q^{\frac{t}{2}\|\vec{n}\|^2+\vec{b}\cdot\vec{n}} \tag{5}$$

The exponent

$$\frac{t}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n}$$

is a degree 2 polynomial of t variables $n_0,\dots,n_{t-1}$. By a change of variables, we will transform it into a quadratic form, i.e. a homogenous polynomial of degree 2.

4.1 The case $5n+4$

Consider the 5 partitions of 4,

$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1.$$

They are all 5-cores and their extended 5-residue diagrams are shown in Figure 13 with region -1 coloured white, region 0 red and region 1 green.

We see that

$$\begin{aligned}
\vec{v}_0 &:= \phi_2((1,1,1,1)) = (1,-1,0,0,0)\\
\vec{v}_1 &:= \phi_2((2,1,1)) = (0,1,-1,0,0)\\
\vec{v}_2 &:= \phi_2((3,1)) = (0,0,1,-1,0)\\
\vec{v}_3 &:= \phi_2((4)) = (0,0,0,1,-1)\\
\vec{v}_4 &:= \phi_2((2,2)) = (1,1,0,-1,-1)
\end{aligned}$$

Obviously, $\vec{v}_0,\dots,\vec{v}_3$ are linearly independent over $\mathbb{R}$. Since $\vec{v}_i\cdot\vec{1}=0$ for all i, this means that $\vec{v}_0,\dots,\vec{v}_4$ span the four-dimensional subspace $\{1\}^\perp$ of $\mathbb{R}^5$. Hence, for any $\vec{n}\in\mathbb{Z}^5$ such that $\vec{n}\cdot\vec{1}=0$ there exists $\vec{\alpha}=(\alpha_0,\dots,\alpha_4)\in\mathbb{R}^5$ such that $\vec{n}=\alpha_0\vec{v}_0+\cdots+\alpha_4\vec{v}_4$. In fact, in that case $\vec{\alpha}$ can be chosen in $\mathbb{Z}^5$: Since $\vec{v}_4=\vec{v}_0+2\vec{v}_1+2\vec{v}_2+\vec{v}_3$, we can choose $\alpha_4=0$. Then $\alpha_0=n_0\in\mathbb{Z}$. Thus $\alpha_1=\alpha_0+n_1\in\mathbb{Z}$, $\alpha_2=\alpha_1+n_2\in\mathbb{Z}$, $\alpha_3=-n_4\in\mathbb{Z}$

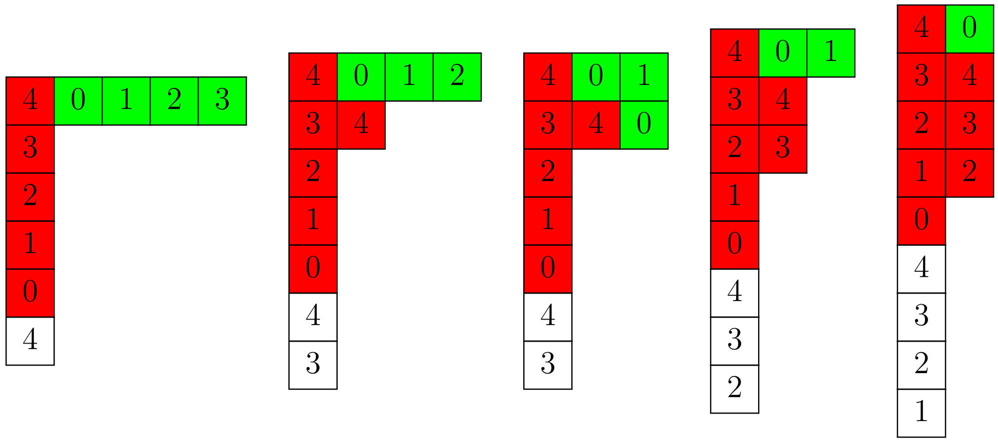

Figure 13: Extended 5-residue diagram of partitions of 4.

Lemma 4.1. *A vector $\vec{n} \in \mathbb{Z}^5$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 4$ mod 5 if, and only if, $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_4 \vec{v}_4$ for some $\vec{\alpha} \in \mathbb{Z}^5$ such that $\vec{\alpha} \cdot \vec{1} = 1$. Further, such an $\vec{\alpha}$ is unique.*

Proof. Indeed, let $\vec{n} \in \mathbb{Z}^5$ such that $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 4$ mod 5. Write $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_4 \vec{v}_4$ with $\vec{\alpha} \in \mathbb{Z}^5$. First we show that $\vec{\alpha} \cdot \vec{1} \equiv 1$ mod 5. Since

$$-\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - 6\alpha_4 = \sum_{i=0}^{4} \alpha_i (\vec{b} \cdot \vec{v}_i) = \vec{b} \cdot \vec{n} \equiv 4 \bmod 5$$

and

$$\vec{b} \cdot \vec{n} + \vec{\alpha} \cdot \vec{1} = -5\alpha_4 \equiv 0 \bmod 5,$$

we conclude that $\vec{\alpha} \cdot \vec{1} \equiv 1$ mod 5. Let $\vec{\beta} = (1, 2, 2, 1, -1)$ and note that $\vec{\beta} \cdot \vec{1} = 5$. Let $k \in \mathbb{Z}$ such that $\vec{\alpha} \cdot \vec{1} = 5k + 1$ and replace $\vec{\alpha}$ by $\vec{\alpha} - k\vec{\beta}$. This does not change $\vec{n}$ because $\vec{v}_4 = \vec{v}_0 + 2\vec{v}_1 + 2\vec{v}_2 + \vec{v}_3$, and we are done.

Conversely, suppose $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_4 \vec{v}_4$ for some $\vec{\alpha} \in \mathbb{Z}^5$ such that $\vec{\alpha} \cdot \vec{1} = 1$. Then since $\vec{v}_i \cdot \vec{1} = 0$ for all i, we have $\vec{n} \cdot \vec{1} = 0$. Further,

$$\vec{b} \cdot \vec{n} = \sum_{i=0}^{4} \alpha_i (\vec{b} \cdot \vec{v}_i) = -\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - 6\alpha_4 = -\vec{\alpha} \cdot \vec{1} - 5\alpha_4 \equiv 4 \bmod 5.$$

Furthermore, having picked $\vec{\alpha} \cdot \vec{1} = 1$ and not $5k + 1$ for any other value of $k \in \mathbb{Z}$, this α is unique, an each such $\vec{\alpha}$ clearly gives a unique $\vec{n}$ □

Now given $\vec{n}$, let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_4)$ be as in Lemma 4.1. Then

$$\vec{n} = (\alpha_0 + \alpha_4, -\alpha_0 + \alpha_1 + \alpha_4, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3 - \alpha_4, -\alpha_3 - \alpha_4)$$

and thus

$$\begin{aligned}
\|\vec{n}\|^2 &= (\alpha_0+\alpha_4)^2 + (-\alpha_0+\alpha_1+\alpha_4)^2 + (-\alpha_1+\alpha_2)^2 + (-\alpha_2+\alpha_3-\alpha_4)^2 + (-\alpha_3-\alpha_4)^2 \\
&= 2\alpha_0^2 + 2\alpha_1^2 + 2\alpha_2^2 + 2\alpha_3^2 + 4\alpha_4^2 - 2\alpha_0\alpha_1 + 2\alpha_1\alpha_4 - 2\alpha_1\alpha_2 - 2\alpha_2\alpha_3 + 2\alpha_2\alpha_4
\end{aligned}$$

Since

$$\vec{b}\cdot\vec{n} = -\alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - 6\alpha_4 = -5\alpha_4 - 1,$$

we obtain

$$\begin{aligned}
\frac{5}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n} &= 5\|\vec{\alpha}\|^2 + 5\alpha_4^2 - 5(\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3) + 5\alpha_1\alpha_4 + 5\alpha_2\alpha_4 - 5\alpha_4 - 1 \\
&= 5\|\vec{\alpha}\|^2 - 5(\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3) + 5\alpha_4(\alpha_1 + \alpha_2 + \alpha_4 - 1) - 1 \\
&= 5\|\vec{\alpha}\|^2 - 5(\alpha_0\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_3) + 5\alpha_4(-\alpha_0 - \alpha_3) - 1 \\
&= 5\|\vec{\alpha}\|^2 - 5\left(\sum_{i=0}^{4}\alpha_i\alpha_{i+1}\right) - 1
\end{aligned}$$

where the indices in the sum are taken mod 5. Define

$$Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - \sum_{i=0}^{4}\alpha_i\alpha_{i+1}.$$

Note that by the above, Equation (5) for $(t,r) = (5,4)$ becomes

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1 \\ \vec{\alpha}\in\mathbb{Z}^5}} q^{5Q(\vec{\alpha})-1} \tag{6}$$

and by multiplication by q followed by the change of variables $q \leftrightarrow q^5$ in Equation (6) we obtain:

$$\sum_{n=0}^{\infty} p(5n+4)q^{n+1} = \frac{1}{(q;q)_\infty^5} \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1 \\ \vec{\alpha}\in\mathbb{Z}^5}} q^{Q(\vec{\alpha})}. \tag{7}$$

By a similar manipulation of the identity

$$\sum_{n=0}^{\infty} p(5n+4)q^{5n+4} = \frac{1}{(q^5;q^5)_\infty^5} \sum_{n=0}^{\infty} a_5(5n+4)q^{5n+4}$$

one finds

$$\sum_{n=0}^{\infty} a_5(5n+4)q^{n+1} = \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1 \\ \vec{\alpha}\in\mathbb{Z}^5}} q^{Q(\vec{\alpha})}. \tag{8}$$

Now $Q(\vec{\alpha})$ is a quadratic form in the variables $(\alpha_0, \ldots, \alpha_4)$. Its automorphism group (i.e. group G of permutations σ of its coordinates such that $Q(\sigma(\vec{\alpha})) = Q(\vec{\alpha})$) is the dihedral group D_5. Indeed, it contains the permutations

$$r := (\alpha_0 \ \alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4)$$

and

$$s = (\alpha_1 \ \alpha_4)(\alpha_2 \ \alpha_3).$$

We have that r is of order 5 and s of order 2. Further, $srs^{-1} = r^{-1}$. Thus the subgroup generated by r and s is isomorphic to D_5 and hence $D_5 \subseteq G$. This means that G contains all the 5-cycles and all the double transpositions. We want to show that $G = D_5$. In fact we can prove the more general Theorem 4.2:

Theorem 4.2. *For t an odd prime, the automorphism group of the quadratic form*

$$Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - \sum_{i=0}^{t-1} \alpha_i \alpha_{i+1}$$

(where the indices in the sum are taken mod t) is isomorphic to the dihedral group D_t.

Proof. Suppose τ is a permutation on $\{\alpha_0, \ldots, \alpha_{t-1}\}$ with σ the permutation it induces on the indices (i.e. $\sigma(i)$ is defined such that $\tau(\alpha_i) = \alpha_{\sigma(i)}$), such that $Q(\vec{\alpha}) = Q(\tau(\vec{\alpha}))$, or equivalently,

$$Q(\alpha_0, \ldots, \alpha_{t-1}) = Q(\alpha_{\sigma(0)}, \ldots, \alpha_{\sigma(t-1)}).$$

Then

$$\sum_{i=0}^{t-1} \alpha_i \alpha_{i+1} = \sum_{i=0}^{t-1} \alpha_{\sigma(i)} \alpha_{\sigma(i+1)}.$$

Since this holds for all $\vec{\alpha}$, $\sigma(i+1) = \sigma(i) \pm 1$ for all i. Suppose σ has a fixed point. Without loss of generality, let that fixed point be 0. Then $\sigma(1) = 1$ or $\sigma(1) = -1 = t-1$. If $\sigma(1) = 1$, then $\sigma(2) = 2$ or 0, but since $\sigma(0) = 0$ we must have $\sigma(2) = 2$. Going on like this we quickly see that $\sigma(i) = i$ for all i. If $\sigma(1) = t-1$, then $\sigma(t-1) = 1$, since $\sigma(1) = \sigma(0) \pm 1 = \pm 1$, and it can't be $-1 = t-1 = \sigma(1)$, so it must be 1. Now $\sigma(2) = \sigma(1) \pm 1 \in \{t-2, 0\}$. Since $\sigma(0) = 0$, we can't have $\sigma(2) = 0$, thus $\sigma(2) = t-2$. Similarly, $\sigma(t-2) = 2$, and in general $\sigma(i) = \sigma(t-i)$ and $\sigma(t-i) = i$ for $i = 1, \ldots, \frac{t-1}{2}$. Thus σ is a $\frac{t-1}{2}$-fold transposition

$$\sigma = (1 \ \ t-1) \cdots \left(\frac{t-1}{2} \ \ \frac{t+1}{2} \right)$$

which is an element of D_t.

Now we only need to show that if σ has no fixed points, then it is a power of the t-cycle

$$(0 \ 1 \ \cdots \ t-1).$$

We have $\sigma(1) = \sigma(0) \pm 1$. Suppose $\sigma(1) = \sigma(0) + 1$. Then $\sigma(2) = \sigma(1) \pm 1 = \sigma(0) + 2$ or $\sigma(0)$. It can't be $\sigma(0)$, thus $\sigma(2) = \sigma(1) + 1$. Similarly, for all i,

$$\sigma(i) = \sigma(i-1) + 1 = \cdots = \sigma(0) + i.$$

Thus $\sigma(\sigma(0)) = 2\sigma(0)$, and in general, $\sigma^n(0) = n\sigma(0)$ where the exponent n denotes n-fold composition (everything is taken mod p). Since $\sigma(0) \neq 0$ and t is prime, $\mathbb{Z}/t\mathbb{Z} = \langle \sigma(0) \rangle$ and thus

$$\sigma = (0\ \sigma(0)\ \sigma(\sigma(0))\ \cdots \sigma^{t-1}(0)) = (0\ 1\ \cdots\ t-1)^{\sigma(0)}.$$

A similar argument works if $\sigma(1) = \sigma(0) - 1$.

We conclude that the automorphism group is contained in the group

$$\left\langle (0\ 1\ \cdots\ t-1), (1\ \ t-1) \cdots \left(\frac{t-1}{2}\ \ \frac{t+1}{2} \right) \right\rangle \cong D_t,$$

and to show the other inclusion, the argument for the case $t = 5$ generalizes easily. $\square$

Returning to the case $t = 5$, let λ be a partition with $\phi_1(\lambda) = (\tilde{\lambda}, \lambda_0, \ldots, \lambda_4)$ for $t = 5$. Denote $\phi(\lambda) = (\alpha(\tilde{\lambda}), \lambda_0, \ldots, \lambda_4)$ where $\alpha(\tilde{\lambda})$ is the vector $\vec{\alpha}$ calculated from $\vec{n} = \phi_2(\tilde{\lambda})$ as above. Then ϕ is a bijection from the set of partitions of $5n+4$ onto its image. Define $\mathrm{crank}'(\lambda) = \vec{b} \cdot \alpha(\tilde{\lambda})$. It has the property of incresing by 1 mod 5 every time the coordinates of $\alpha(\tilde{\lambda})$ are cyclically permuted: Indeed, since

$$\vec{b} \cdot (\alpha_0, \ldots, \alpha_4) = 0 \cdot \alpha_0 + 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + 3 \cdot \alpha_3 + 4 \cdot \alpha_4,$$

permuting the coordinates once cyclically yields

$$0 \cdot \alpha_4 + 1 \cdot \alpha_0 + 2 \cdot \alpha_1 + 3 \cdot \alpha_2 + 4 \cdot \alpha_3.$$

The difference is

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 - 4\alpha_4 = \vec{\alpha} \cdot \vec{1} - 5\alpha_4 \equiv 1 \bmod 5$$

Since the automorphism group of $Q(\vec{\alpha})$ is D_5, permuting the coordinates of $\alpha(\tilde{\lambda})$ via the 5-cycle $(0\ 1\ 2\ 3\ 4)$ doesn't change the number which λ partitions, i.e.

$$5Q(\alpha(\tilde{\lambda})) - 1 + 5\sum_{i=0}^{4} |\lambda_i| = 5n + 4$$

The operation of permuting the coordinates cyclically has no fixed point $\vec{\alpha}$ with $\vec{\alpha} \cdot \vec{1} = 1$, since a fixed point would have all coordinates equal and therefore all coordinates equal to $1/5$, contradicting the fact that $\vec{\alpha} \in \mathbb{Z}^5$.

It is now clear that the 5-cycle $\sigma = (\alpha_0\ \alpha_1\ \alpha_2\ \alpha_3\ \alpha_4)$ acts on the partitions of $5n+4$ with no fixed points:

$$\sigma \cdot \lambda = (\sigma(\alpha(\tilde{\lambda})), \lambda_0, \ldots, \lambda_4)$$

Where $\alpha(\tilde{\lambda})$ and λ_i are defined as before by the bijections ϕ_1 and ϕ_2, and if $\alpha(\tilde{\lambda}) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, then $\sigma(\alpha(\tilde{\lambda})) = (\alpha_4, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$. Each of its orbits is of cardinality 5 and thus there are $p(5n+4)/5$ orbits, proving the first of Ramanujan's congruences (Theorem 1.1).

Since $\text{crank}'(\lambda)$ increases by 1 mod 5 by each action of the 5-cycle, we have explicit bijections (given by the 5-cycle) between the residue classes of the partitions of $5n+4$.

We still need to modify $\text{crank}'(\lambda)$, for it to satisfy both Dyson's properties of a crank. We need the condition given in Equation (1), and for that it suffices to make the crank switch signs when conjugating. We give a crank in terms of $\vec{n}$ in Theorem 4.3.

Theorem 4.3. *A crank for partitions λ of $5n+4$ is given by the following algorithm.*

(1) Find the 5-core $\tilde{\lambda}$ of λ by the bijection ϕ_1.

(2) Find $\phi_2(\tilde{\lambda}) = \vec{n}$.

(3) Let $\text{crank}(\lambda) = 4n_0 + n_1 + n_3 + 4n_4$ mod 5.

Proof. We have

$$\vec{n} = (\alpha_0 + \alpha_4, -\alpha_0 + \alpha_1 + \alpha_4, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3 - \alpha_4, -\alpha_3 - \alpha_4)$$

yielding the system of equations

$$\begin{cases} n_0 &= \alpha_0 + \alpha_4 \\ n_1 &= -\alpha_0 + \alpha_1 + \alpha_4 \\ n_2 &= -\alpha_1 + \alpha_2 \\ n_3 &= -\alpha_2 + \alpha_3 - \alpha_4 \\ n_4 &= -\alpha_3 - \alpha_4 \end{cases}$$

Thus, modulo 5 we have the following

$$\begin{aligned} 4n_0 + n_1 + n_3 + 4n_4 &= 4(\alpha_0 + \alpha_4) - \alpha_0 + \alpha_1 + \alpha_4 - \alpha_2 + \alpha_3 - \alpha_4 - 4(\alpha_3 + \alpha_4) \\ &= 3\alpha_0 + \alpha_1 - \alpha_2 - 3\alpha_3 \\ &= 3\vec{\alpha} \cdot \vec{1} + 3\alpha_1 + \alpha_2 + 4\alpha_3 + 2\alpha_4 \\ &= 3 + 3\vec{b} \cdot \vec{\alpha} \end{aligned}$$

Since the map $\mathbb{Z}/5\mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$, $x \mapsto 3 + 3x$ is bijective (its inverse is $x \mapsto 4 + 2x$) and since $\vec{b} \cdot \vec{\alpha}$ splits the partitions into 5 equinumerous classes, so does $\text{crank}(\lambda)$.

Since $\phi_2(\lambda') = (-n_{t-1}, \ldots, -n_0)$ (cf. Remark 2.5), we have $\text{crank}(\lambda') = -\text{crank}(\lambda)$, and we conclude that $\text{crank}(\lambda)$ is indeed a crank. □

4.2 The cases $7n+5$ and $11n+6$

We now seek to make the analogous construction for partitions of $7n+5$ and $11n+6$. For the seven partitions of 5 (all of which are 7-cores), we have

$$\begin{aligned}
\vec{v}_0 &:= \phi_2((5)) = (0,0,0,0,1,0,-1)\\
\vec{v}_1 &:= \phi_2((4,1)) = (0,0,0,1,0,-1,0)\\
\vec{v}_2 &:= \phi_2((3,2)) = (1,0,1,0,0,-1,-1)\\
\vec{v}_3 &:= \phi_2((3,1,1)) = (0,0,1,0,-1,0,0)\\
\vec{v}_4 &:= \phi_2((2,2,1)) = (1,1,0,0,-1,0,-1)\\
\vec{v}_5 &:= \phi_2((2,1,1,1)) = (0,1,0,-1,0,0,0)\\
\vec{v}_6 &:= \phi_2((1,1,1,1,1)) = (1,0,-1,0,0,0,0)
\end{aligned}$$

The vectors $\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5, \vec{v}_6$ are easily seen to be linearly independent. Thus the vectors $\vec{v}_i$ span the 6 dimensional space $\{1\}^{\perp}$. We want to prove an analogue of Lemma 4.1:

Lemma 4.4. *A vector $\vec{n} \in \mathbb{Z}^7$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 5 \bmod 7$ if, and only if, $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_6 \vec{v}_6$ for some $\vec{\alpha} \in \mathbb{Z}^7$ such that $\vec{\alpha} \cdot \vec{1} = 1$. Further, such an $\vec{\alpha}$ is unique.*

Proof. Suppose first that $\vec{n} \in \mathbb{Z}^7$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 5 \bmod 7$. We know that there exists $\vec{\alpha} \in \mathbb{R}^7$ such that $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_6 \vec{v}_6$. In fact, such an $\vec{\alpha}$ belongs to $\mathbb{Z}^7$: Since

$$\vec{v}_4 = 2\vec{v}_0 + \vec{v}_1 - \vec{v}_2 + 3\vec{v}_3 + \vec{v}_5 + 2\vec{v}_6$$

we can choose $\alpha_4 = 0$. Then as a column vector,

$$\vec{n} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = \begin{pmatrix} \alpha_2 + \alpha_6 \\ \alpha_5 \\ \alpha_2 + \alpha_3 - \alpha_6 \\ \alpha_1 - \alpha_5 \\ \alpha_0 - \alpha_3 \\ -\alpha_1 - \alpha_2 \\ -\alpha_0 - \alpha_2 \end{pmatrix},$$

thus $\alpha_5 = n_1 \in \mathbb{Z}$, $\alpha_1 = n_3 + \alpha_5 \in \mathbb{Z}$, $\alpha_2 = -n_5 - \alpha_1 \in \mathbb{Z}$, $\alpha_0 = -n_6 - \alpha_2 \in \mathbb{Z}$, $\alpha_3 = \alpha_0 - n_4 \in \mathbb{Z}$ and $\alpha_6 = n_0 - \alpha_2 \in \mathbb{Z}$.

Now we show that $\vec{\alpha} \cdot \vec{1} \equiv 1 \bmod 7$. Since

$$-2\alpha_0 - 2\alpha_1 - 9\alpha_2 - 2\alpha_3 - 2\alpha_5 - 2\alpha_6 = \vec{b} \cdot \vec{n} \equiv 5 \bmod 7$$

and

$$2\vec{\alpha} \cdot \vec{1} + \vec{b} \cdot \vec{n} = -7\alpha_2 \equiv 0 \bmod 7$$

we conclude that

$$2\vec{\alpha} \cdot 1 \equiv 2 \bmod 7$$

and thus since 2 and 7 are coprime,

$$\vec{\alpha} \cdot 1 \equiv 1 \bmod 7.$$

Now let $\vec{\beta} = (2, 1, -1, 3, -1, 1, 2)$. Note that $\vec{\beta} \cdot \vec{1} = 7 \equiv 0 \bmod 7$ and replacing $\vec{\alpha}$ by $\vec{\alpha} - k\vec{\beta}$ where $k \in \mathbb{Z}$ such that $\vec{\alpha} \cdot \vec{1} = 7k + 1$ doesn't change anything, and we're done.

Conversely, suppose $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_6 \vec{v}_6$ for some $\vec{\alpha} \in \mathbb{Z}^7$ such that $\vec{\alpha} \cdot \vec{1} = 1$. Write $\vec{\alpha} = \vec{\alpha'} - \alpha_4 \vec{\beta}$ with $\alpha'_4 = 0$. Then $\vec{n} = \alpha'_0 \vec{v}_0 + \cdots \alpha'_6 \vec{v}_6$, $\vec{\alpha'} \cdot \vec{1} \equiv 1 \bmod 7$ and

$$\vec{b} \cdot \vec{n} = -2\alpha'_0 - 2\alpha'_1 - 9\alpha'_2 - 2\alpha'_3 - 2\alpha'_5 - 2\alpha'_6 = -2\vec{\alpha'} \cdot \vec{1} - 7\alpha'_2 \equiv 5 \bmod 7.$$

Clearly, $\vec{n} \cdot \vec{1} = 0$, and we are done. $\square$

Given $\vec{n}$, let $\vec{\alpha} = (\alpha_0, \ldots, \alpha_6)$ be as in Lemma 4.4. Then

$$\vec{n} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = \begin{pmatrix} \alpha_2 + \alpha_6 \\ \alpha_5 \\ \alpha_2 + \alpha_3 - \alpha_6 \\ \alpha_1 - \alpha_5 \\ \alpha_0 - \alpha_3 \\ -\alpha_1 - \alpha_2 \\ -\alpha_0 - \alpha_2 \end{pmatrix} + \alpha_4 \vec{v}_4 = \begin{pmatrix} \alpha_2 + \alpha_4 + \alpha_6 \\ \alpha_4 + \alpha_5 \\ \alpha_2 + \alpha_3 - \alpha_6 \\ \alpha_1 - \alpha_5 \\ \alpha_0 - \alpha_3 - \alpha_4 \\ -\alpha_1 - \alpha_2 \\ -\alpha_0 - \alpha_2 - \alpha_4 \end{pmatrix}$$

and thus

$$\begin{aligned} \|\vec{n}\|^2 &= (\alpha_2 + \alpha_4 + \alpha_6)^2 + (\alpha_4 + \alpha_5)^2 + (\alpha_2 + \alpha_3 - \alpha_6)^2 + (\alpha_1 - \alpha_5)^2 \\ &\quad + (\alpha_0 - \alpha_3 - \alpha_4)^2 + (-\alpha_1 - \alpha_2)^2 + (-\alpha_0 - \alpha_2 - \alpha_4)^2 \\ &= 2\alpha_0^2 + 2\alpha_1^2 + 4\alpha_2^2 + 2\alpha_3^2 + 4\alpha_4^2 + 2\alpha_5^2 + 2\alpha_6^2 \\ &\quad + 2\alpha_0\alpha_2 - 2\alpha_0\alpha_3 + 2\alpha_1\alpha_2 - 2\alpha_1\alpha_5 + 2\alpha_2\alpha_3 \\ &\quad + 4\alpha_2\alpha_4 + 2\alpha_3\alpha_4 - 2\alpha_3\alpha_6 + 2\alpha_4\alpha_5 + 2\alpha_4\alpha_6 \end{aligned}$$

and

$$\begin{aligned} \vec{b} \cdot \vec{n} &= (\alpha_4 + \alpha_5) + 2(\alpha_2 + \alpha_3 - \alpha_6) + 3(\alpha_1 - \alpha_5) \\ &\quad + 4(\alpha_0 - \alpha_3 - \alpha_4) + 5(-\alpha_1 - \alpha_2) + 6(-\alpha_0 - \alpha_2 - \alpha_4) \\ &= -2\alpha_0 - 2\alpha_1 - 9\alpha_2 - 2\alpha_3 - 9\alpha_4 - 2\alpha_5 - 2\alpha_6 \\ &= -2\vec{\alpha} \cdot \vec{1} - 7\alpha_2 - 7\alpha_4 = -2 - 7\alpha_2 - 7\alpha_4 \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{7}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n} &= 7\|\vec{\alpha}\|^2 - 7(\alpha_0\alpha_3 + \alpha_3\alpha_6 + \alpha_1\alpha_5) - 2 \\ &\quad - 7\alpha_2(1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \\ &\quad - 7\alpha_4(1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6) \\ &= 7\|\vec{\alpha}\|^2 - 7(\alpha_0\alpha_3 + \alpha_3\alpha_6 + \alpha_1\alpha_5) \\ &\quad - 7\alpha_2(\alpha_5 + \alpha_6) - 7\alpha_4(\alpha_0 + \alpha_1) - 2 \\ &= 7\|\vec{\alpha}\|^2 - 7(\alpha_0\alpha_3 + \alpha_3\alpha_6 + \alpha_6\alpha_2 + \alpha_2\alpha_5 + \alpha_5\alpha_1 + \alpha_1\alpha_4 + \alpha_4\alpha_0) - 2\end{aligned}$$

By renaming the coordinates of $\vec{\alpha}$ (or initially the vectors $\vec{v}_i$) in the following way:

$$\begin{aligned}\alpha_0 &\to \alpha_0 \\ \alpha_3 &\to \alpha_1 \\ \alpha_6 &\to \alpha_2 \\ \alpha_2 &\to \alpha_3 \\ \alpha_5 &\to \alpha_4 \\ \alpha_1 &\to \alpha_5 \\ \alpha_4 &\to \alpha_6,\end{aligned}$$

one obtains

$$\frac{7}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n} = 7Q(\vec{\alpha}) - 2$$

where

$$Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - \sum_{i=0}^{6}\alpha_i\alpha_{i+1}$$

where the indices in the sum are taken mod 7. In the same way as before, one finds the analogues of Equations (7) and (8):

$$\sum_{n=0}^{\infty} p(7n+5)q^{n+1} = \frac{1}{(q;q)_\infty^7}\sum_{\substack{\vec{\alpha}\cdot\vec{1}=1\\ \vec{\alpha}\in\mathbb{Z}^7}} q^{Q(\vec{\alpha})} \tag{9}$$

$$\sum_{n=0}^{\infty} a_7(7n+5)q^{n+1} = \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1\\ \vec{\alpha}\in\mathbb{Z}^7}} q^{Q(\vec{\alpha})}. \tag{10}$$

By Theorem 4.2, the dihedral group D_7 (generated by the 7-cycle (0 1 2 3 4 5 6) and the triple transposition (1 6)(2 5)(3 4) in S_7) is the automorphism group of Q, and similarly a statistic splitting up the partitions into 7 equinumerous classes is given by $\vec{b}\cdot\vec{\alpha}$.

Theorem 4.5. *A crank for partitions λ of $7n+5$ is given by the following algorithm.*

(1) Find the 7-core $\tilde{\lambda}$ of λ by the bijection ϕ_1.

(2) Find $\phi_2(\tilde{\lambda}) = \vec{n}$.

(3) Let $\operatorname{crank}(\lambda) = 4n_0 + 2n_1 + n_2 + n_4 + 2n_5 + 4n_6$ *mod* 7.

Proof. With the renaming of the $\vec{\alpha}$ coordinates, we have

$$\vec{n} = \begin{pmatrix} \alpha_2 + \alpha_3 + \alpha_6 \\ \alpha_4 + \alpha_6 \\ \alpha_1 - \alpha_2 + \alpha_3 \\ -\alpha_4 + \alpha_5 \\ \alpha_0 - \alpha_1 - \alpha_6 \\ -\alpha_3 - \alpha_5 \\ -\alpha_0 - \alpha_3 - \alpha_6 \end{pmatrix}$$

and

$$\vec{b} \cdot \vec{n} = -2 - 7\alpha_3 - 7\alpha_6.$$

Thus, modulo 7 we have the following:

$$\begin{aligned} 4n_0 + 2n_1 + n_2 + n_4 + 2n_5 + 4n_6 &= 4(\alpha_2 + \alpha_3 + \alpha_6) + 2(\alpha_4 + \alpha_6) + \alpha_1 - \alpha_2 + \alpha_3 \\ &\quad + \alpha_0 - \alpha_1 - \alpha_6 + 2(-\alpha_3 - \alpha_5) + 4(-\alpha_0 - \alpha_3 - \alpha_6) \\ &= -3\alpha_0 + 3\alpha_2 - \alpha_3 + 2\alpha_4 - 2\alpha_5 + \alpha_6 \\ &= 4\vec{\alpha} \cdot \vec{1} + 3\alpha_1 + 6\alpha_2 + 2\alpha_3 + 5\alpha_4 + \alpha_5 + 4\alpha_6 \\ &= 4 + 3\vec{b} \cdot \vec{\alpha}. \end{aligned}$$

The map $\mathbb{Z}/7\mathbb{Z} \to \mathbb{Z}/7\mathbb{Z}$, $x \mapsto 4+3x$ is a bijection (its inverse is $x \mapsto 5x+1$). Since $\vec{b} \cdot \vec{\alpha}$ splits the partitions into 7 equinumerous classes and clearly $\operatorname{crank}(\lambda') = -\operatorname{crank}(\lambda)$, this yields the desired result. □

For the eleven paritions of 6 (all of which are 11-cores) we now have

$$\begin{aligned} \vec{v}_0 &:= \phi_2((6)) = (0,0,0,0,0,1,0,0,0,0,-1) \\ \vec{v}_1 &:= \phi_2((5,1)) = (0,0,0,0,1,0,0,0,0,-1,0) \\ \vec{v}_2 &:= \phi_2((4,2)) = (1,0,0,1,0,0,0,0,0,-1,-1) \\ \vec{v}_3 &:= \phi_2((4,1,1)) = (0,0,0,1,0,0,0,0,-1,0,0) \\ \vec{v}_4 &:= \phi_2((3,3)) = (0,1,1,0,0,0,0,0,0,-1,-1) \\ \vec{v}_5 &:= \phi_2((3,2,1)) = (1,0,1,0,0,0,0,0,-1,0,-1) \\ \vec{v}_6 &:= \phi_2((3,1,1,1)) = (0,0,1,0,0,0,0,-1,0,0,0) \\ \vec{v}_7 &:= \phi_2((2,2,2)) = (1,1,0,0,0,0,0,0,-1,-1,0) \\ \vec{v}_8 &:= \phi_2((2,2,1,1)) = (1,1,0,0,0,0,0,-1,0,0,-1) \\ \vec{v}_9 &:= \phi_2((2,1,1,1,1)) = (0,1,0,0,0,0,-1,0,0,0,0) \\ \vec{v}_{10} &:= \phi_2((1,1,1,1,1,1)) = (1,0,0,0,0,-1,0,0,0,0,0) \end{aligned}$$

Now we obtain

$$\begin{aligned}
\vec{u}_0 &:= (1,-1,0,0,0,0,0,0,0,0,0) = -\vec{v}_0 + \vec{v}_2 - \vec{v}_3 - \vec{v}_4 + \vec{v}_5 - \vec{v}_{10} \\
\vec{u}_1 &:= (0,1,-1,0,0,0,0,0,0,0,0) = -\vec{v}_0 - \vec{v}_6 + \vec{v}_8 - \vec{v}_{10} \\
\vec{u}_2 &:= (0,0,1,-1,0,0,0,0,0,0,0) = -\vec{v}_0 - \vec{v}_3 + \vec{v}_5 - \vec{v}_{10} \\
\vec{u}_3 &:= (0,0,0,1,-1,0,0,0,0,0,0) = -\vec{v}_0 - \vec{v}_1 + \vec{v}_2 \\
\vec{u}_4 &:= (0,0,0,0,1,-1,0,0,0,0,0) = -2\vec{v}_0 + \vec{v}_1 + \vec{v}_5 - \vec{v}_6 - \vec{v}_7 + \vec{v}_8 - \vec{v}_{10} \\
\vec{u}_5 &:= (0,0,0,0,0,1,-1,0,0,0,0) = -\vec{v}_0 - \vec{v}_2 - \vec{v}_3 - \vec{v}_4 + \vec{v}_5 + \vec{v}_9 - 2\vec{v}_{10} \\
\vec{u}_6 &:= (0,0,0,0,0,0,1,-1,0,0,0) = -\vec{v}_0 + \vec{v}_8 - \vec{v}_9 - \vec{v}_{10} \\
\vec{u}_7 &:= (0,0,0,0,0,0,0,1,-1,0,0) = -\vec{v}_0 + \vec{v}_5 - \vec{v}_6 - \vec{v}_{10} \\
\vec{u}_8 &:= (0,0,0,0,0,0,0,0,1,-1,0) = -\vec{v}_0 + \vec{v}_2 - \vec{v}_3 - \vec{v}_{10} \\
\vec{u}_9 &:= (0,0,0,0,0,0,0,0,0,1,-1) = -\vec{v}_0 + \vec{v}_5 - \vec{v}_6 - \vec{v}_7 + \vec{v}_8 - \vec{v}_{10}.
\end{aligned}$$

The vectors $\vec{u}_i$ obviously form a basis of $\{1\}^\perp$ in $\mathbb{R}^{11}$ (and by the above, the vectors $\vec{v}_i$ span that same subspace). Let $\vec{n} \in \mathbb{Z}^{11}$ such that $\vec{n} \cdot \vec{1} = 0$. The unique coefficients $\beta_i \in \mathbb{R}$ of the linear combination $\vec{n} = \beta_0\vec{u}_0 + \cdots + \beta_9\vec{u}_9$ are integers, since $\beta_0 = n_0 \in \mathbb{Z}$, $\beta_1 = n_1 + \beta_0 \in \mathbb{Z}$, etc. Thus if $\vec{n} = \alpha_0\vec{v}_0 + \cdots + \alpha_{10}\vec{v}_{10}$ then $\vec{\alpha} \in \mathbb{Z}^{11}$, since the $\vec{u}_i$'s are integer linear combinations of the $\vec{v}_i$'s.

As usual, we have

Lemma 4.6. *A vector $\vec{n} \in \mathbb{Z}^{11}$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 6$ mod 11 if, and only if, $\vec{n} = \alpha_0\vec{v}_0 + \cdots + \alpha_{10}\vec{v}_{10}$ for some $\vec{\alpha} \in \mathbb{Z}^{11}$ such that $\vec{\alpha} \cdot \vec{1} = 1$. Further, such an $\vec{\alpha}$ is unique.*

Proof. Suppose $\vec{n} \in \mathbb{Z}^{11}$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 6$ mod 11. Let $\vec{\alpha} \in \mathbb{Z}^{11}$ such that $\vec{n} = \alpha_0\vec{v}_0 + \cdots + \alpha_{10}\vec{v}_{10}$. We begin by showing that $\vec{\alpha} \cdot \vec{1} \equiv 1$ mod 11. We have

$$\vec{n} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ n_8 \\ n_9 \\ n_{10} \end{pmatrix} = \begin{pmatrix} \alpha_2 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_{10} \\ \alpha_4 + \alpha_7 + \alpha_8 + \alpha_9 \\ \alpha_4 + \alpha_5 + \alpha_6 \\ \alpha_2 + \alpha_3 \\ \alpha_1 \\ \alpha_0 - \alpha_{10} \\ -\alpha_9 \\ -\alpha_6 - \alpha_8 \\ -\alpha_3 - \alpha_5 - \alpha_7 \\ -\alpha_1 - \alpha_2 - \alpha_4 - \alpha_7 \\ -\alpha_0 - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_8 \end{pmatrix}$$

and

$$\begin{aligned}
&- 5\alpha_0 - 5\alpha_1 - 16\alpha_2 - 5\alpha_3 - 16\alpha_4 - 16\alpha_5 - 5\alpha_6 - 16\alpha_7 - 16\alpha_8 - 5\alpha_9 - 5\alpha_{10} \\
&= \vec{b} \cdot \vec{n} \\
&\equiv 6 \bmod 11.
\end{aligned}$$

Therefore,

$$5\vec{\alpha}\cdot\vec{1} \equiv 5 \bmod 11,$$

and multiplying both sides by 9, one obtains

$$\vec{\alpha}\cdot\vec{1} \equiv 1 \bmod 11.$$

Note that

$$\vec{v}_7 = -6\vec{v}_0 + 2\vec{v}_2 - 2\vec{v}_3 - \vec{v}_4 + 3\vec{v}_5 - 2\vec{v}_6 + 2\vec{v}_8 - 6\vec{v}_{10}.$$

Let $k \in \mathbb{Z}$ such that $\vec{\alpha}\cdot\vec{1} = 11k + 1$. Let

$$\vec{\beta} = (-6, 0, 2, -2, -1, 3, -2, -1, 2, 0, -6).$$

Since $\vec{\beta}\cdot\vec{1} = -11$ and adding or subtracting $\vec{\beta}$ from $\vec{\alpha}$ doesn't change $\vec{n}$, we can replace $\vec{\alpha}$ by $\vec{\alpha} + k\vec{\beta}$.

For the converse, note that

$$\begin{aligned}\vec{b}\cdot\vec{n} &= -5\vec{\alpha}\cdot\vec{1} - 11(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8)\\ &= -5 - 11(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_7 + \alpha_8)\\ &\equiv 6 \bmod 11.\end{aligned}$$

□

Given $\vec{n}$, let $\vec{\alpha}$ be as in Lemma 4.6. Then

$$\begin{aligned}\|\vec{n}\|^2 &= (\alpha_2 + \alpha_5 + \alpha_7 + \alpha_8 + \alpha_{10})^2 + (\alpha_4 + \alpha_7 + \alpha_8 + \alpha_9)^2 + (\alpha_4 + \alpha_5 + \alpha_6)^2\\ &\quad + (\alpha_2 + \alpha_3)^2 + \alpha_1^2 + (\alpha_0 - \alpha_{10})^2 + (-\alpha_9)^2 + (-\alpha_6 - \alpha_8)^2\\ &\quad + (-\alpha_3 - \alpha_5 - \alpha_7)^2 + (-\alpha_1 - \alpha_2 - \alpha_4 - \alpha_7)^2 + (-\alpha_0 - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_8)^2\\ &= 2\alpha_0^2 + 2\alpha_1^2 + 4\alpha_2^2 + 2\alpha_3^2 + 4\alpha_4^2 + 4\alpha_5^2 + 2\alpha_6^2 + 4\alpha_7^2 + 4\alpha_8^2 + 2\alpha_9^2 + 2\alpha_{10}^2\\ &\quad + 2\alpha_0\alpha_2 + 2\alpha_0\alpha_4 + 2\alpha_0\alpha_5 + 2\alpha_0\alpha_8 - 2\alpha_0\alpha_{10} + 2\alpha_1\alpha_2 + 2\alpha_1\alpha_4 + 2\alpha_1\alpha_7\\ &\quad + 2\alpha_2\alpha_3 + 4\alpha_2\alpha_4 + 4\alpha_2\alpha_5 + 4\alpha_2\alpha_7 + 4\alpha_2\alpha_8 + 2\alpha_2\alpha_{10} + 2\alpha_3\alpha_5 + 2\alpha_3\alpha_7\\ &\quad + 4\alpha_4\alpha_5 + 2\alpha_4\alpha_6 + 4\alpha_4\alpha_7 + 4\alpha_4\alpha_8 + 2\alpha_4\alpha_9 + 2\alpha_5\alpha_6 + 4\alpha_5\alpha_7 + 4\alpha_5\alpha_8\\ &\quad + 2\alpha_5\alpha_{10} + 2\alpha_6\alpha_8 + 4\alpha_7\alpha_8 + 2\alpha_7\alpha_9 + 2\alpha_7\alpha_{10} + 2\alpha_8\alpha_9 + 2\alpha_8\alpha_{10}\end{aligned}$$

and

$$\vec{b}\cdot\vec{n} = -5 - 11\alpha_2 - 11\alpha_4 - 11\alpha_5 - 11\alpha_7 - 11\alpha_8.$$

Therefore,

$$\begin{aligned}
\frac{11}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n} &= 11\|\vec{\alpha}\|^2 - 11\alpha_0\alpha_{10} - 5 \\
&\quad - 11\alpha_2(1-\alpha_0-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_7-\alpha_8-\alpha_{10}) \\
&\quad - 11\alpha_4(1-\alpha_0-\alpha_1-\alpha_2-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9) \\
&\quad - 11\alpha_5(1-\alpha_0-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_{10}) \\
&\quad - 11\alpha_7(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}) \\
&\quad - 11\alpha_8(1-\alpha_0-\alpha_2-\alpha_4-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9-\alpha_{10}) \\
&= 11\|\vec{\alpha}\|^2 - 11\alpha_0\alpha_{10} - 5 - 11\alpha_2(\alpha_6+\alpha_9) - 11\alpha_4(\alpha_3+\alpha_{10}) \\
&\quad - 11\alpha_5(\alpha_1+\alpha_9) - 11\alpha_7(\alpha_0+\alpha_6) - 11\alpha_8(\alpha_1+\alpha_3) \\
&= 11\|\vec{\alpha}\|^2 - 11(\alpha_0\alpha_{10}+\alpha_{10}\alpha_4+\alpha_4\alpha_3+\alpha_3\alpha_8+\alpha_8\alpha_1+\alpha_1\alpha_5 \\
&\quad + \alpha_5\alpha_9+\alpha_9\alpha_2+\alpha_2\alpha_6+\alpha_6\alpha_7+\alpha_7\alpha_0) - 5
\end{aligned}$$

As before, we can make the following change of variables:

$$\begin{aligned}
\alpha_0 &\to \alpha_0 \\
\alpha_{10} &\to \alpha_1 \\
\alpha_4 &\to \alpha_2 \\
\alpha_3 &\to \alpha_3 \\
\alpha_8 &\to \alpha_4 \\
\alpha_1 &\to \alpha_5 \\
\alpha_5 &\to \alpha_6 \\
\alpha_9 &\to \alpha_7 \\
\alpha_2 &\to \alpha_8 \\
\alpha_6 &\to \alpha_9 \\
\alpha_7 &\to \alpha_{10}
\end{aligned}$$

and obtain

$$\frac{11}{2}\|\vec{n}\|^2 + \vec{b}\cdot\vec{n} = 11Q(\vec{\alpha}) - 5$$

where

$$Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - \sum_{i=0}^{10}\alpha_i\alpha_{i+1}$$

and the indices in the sum are taken mod 11. As usual, we obtain

$$\sum_{n=0}^{\infty} p(11n+6)q^{n+1} = \frac{1}{(q;q)_\infty^{11}} \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1 \\ \vec{\alpha}\in\mathbb{Z}^{11}}} q^{Q(\vec{\alpha})} \tag{11}$$

$$\sum_{n=0}^{\infty} a_{11}(11n+6)q^{n+1} = \sum_{\substack{\vec{\alpha}\cdot\vec{1}=1\\ \vec{\alpha}\in\mathbb{Z}^{11}}} q^{Q(\vec{\alpha})}. \tag{12}$$

By Theorem 4.2, the automorphism group is isomorphic to the dihedral group D_{11}. Thus as before, a statistic splitting the partitions into 11 equinumerous classes is given by $\vec{b}\cdot\vec{\alpha}$.

Theorem 4.7. *A crank for partitions λ of $11n+6$ is given by the following algorithm.*

(1) Find the 11*-core $\tilde{\lambda}$ of λ by the bijection ϕ_1.*

(2) Find $\phi_2(\tilde{\lambda}) = \vec{n}$.

(3) Let $\operatorname{crank}(\lambda) = 4n_0 + 9n_1 + 5n_2 + 3n_3 + n_4 + n_6 + 3n_7 + 5n_8 + 9n_9 + 4n_{10}$ *mod* 11.

Proof. With the renaming of the $\vec{\alpha}$ coordinates, we have

$$\vec{n} = \begin{pmatrix} \alpha_1+\alpha_4+\alpha_6+\alpha_8+\alpha_{10} \\ \alpha_2+\alpha_4+\alpha_7+\alpha_{10} \\ \alpha_2+\alpha_6+\alpha_9 \\ \alpha_3+\alpha_8 \\ \alpha_5 \\ \alpha_0-\alpha_1 \\ -\alpha_7 \\ -\alpha_4-\alpha_9 \\ -\alpha_3-\alpha_6-\alpha_{10} \\ -\alpha_2-\alpha_5-\alpha_8-\alpha_{10} \\ -\alpha_0-\alpha_2-\alpha_4-\alpha_6-\alpha_8 \end{pmatrix}$$

and

$$\vec{b}\cdot\vec{\alpha} = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 6\alpha_6 + 7\alpha_7 + 8\alpha_8 + 9\alpha_9 + 10\alpha_{10}$$

Thus, modulo 11 we have the following:

$$\begin{aligned}
&4n_0 + 9n_1 + 5n_2 + 3n_3 + n_4 + n_6 + 3n_7 + 5n_8 + 9n_9 + 4n_{10} \\
&= 4(\alpha_1+\alpha_4+\alpha_6+\alpha_8+\alpha_{10}) + 9(\alpha_2+\alpha_4+\alpha_7+\alpha_{10}) + 5(\alpha_2+\alpha_6+\alpha_9) + 3(\alpha_3+\alpha_8) \\
&\quad + \alpha_5 - \alpha_7 + 3(-\alpha_4-\alpha_9) + 5(-\alpha_3-\alpha_6-\alpha_{10}) + 9(-\alpha_2-\alpha_5-\alpha_8-\alpha_{10}) \\
&\quad + 4(-\alpha_0-\alpha_2-\alpha_4-\alpha_6-\alpha_8) \\
&= -4\alpha_0 + 4\alpha_1 + \alpha_2 - 2\alpha_3 + 6\alpha_4 - 8\alpha_5 + 8\alpha_7 - 6\alpha_8 + 2\alpha_9 - \alpha_{10} \\
&= 7\vec{\alpha}\cdot\vec{1} + 8\alpha_1 + 5\alpha_2 + 2\alpha_3 + 10\alpha_4 + 7\alpha_5 + 4\alpha_6 + \alpha_7 + 9\alpha_8 + 6\alpha_9 + 3\alpha_{10} \\
&= 7 + 8\vec{b}\cdot\vec{\alpha}.
\end{aligned}$$

The map $\mathbb{Z}/11\mathbb{Z} \to \mathbb{Z}/11\mathbb{Z}$, $x \mapsto 7+8x$ is a bijection (its inverse is $x \mapsto 7x+6$). Since $\vec{b}\cdot\vec{\alpha}$ splits the partitions into 11 equinumerous classes and $\operatorname{crank}(\lambda') = -\operatorname{crank}(\lambda)$, this yields the desired result. □

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