Cranks and t-cores

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- *Partition* of $n \in \mathbb{N}$: finite sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$ and $n = \lambda_1 + \cdots + \lambda_k$.

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- The λ_i's are called the *parts* of λ; n is denoted $|\lambda|$ and the number of partitions of n is denoted $p(n)$.
- A partition may be represented by a so-called Young diagram, i.e. an array of cells such that the i-th line has λ_i cells for $i = 1, \ldots, k$.

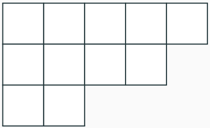

Figure 1: Young diagram of the partition $\lambda = (5, 4, 2)$.

Conjugate partition

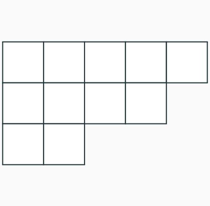

Figure 2: Young diagrams of $\lambda = (5, 4, 2)$ and its conjugate $\lambda' = (3, 3, 2, 2, 1)$.

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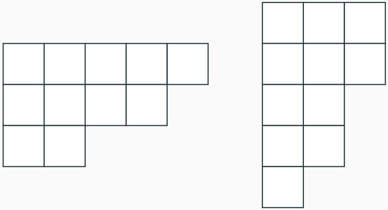

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- Obvious 1-1 correspondence between partitions and Young diagrams.
- *Conjugate partition* of λ: the partition λ' obtained by transposing the Young diagram of λ.

Ramanujan's congruences

Theorem. (Ramanujan's congruences). For $n \in \mathbb{N}$,

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$
$$p(11n+6) \equiv 0 \pmod{11}$$

Dyson's rank

- "Combinatorial" proof of Ramanujan's congruences: find a statistic on partitions of $5n + 4$ (resp. $7n + 5$, $11n + 6$) splitting them into 5 (resp. 7, 11) equinumerous classes.

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- Denote $N(m,t,n) = \#\{\lambda \text{ partition of } n | \mathsf{rank}(\lambda) \equiv m \pmod{t}\}$. Then $N(t-m,t,n) = N(m,t,n)$.
- Dyson's conjecture: rank is the desired statistic for $5n+4$ and $7n+5$, i.e.

$$N(m,5,5n+4) = \frac{p(5n+4)}{5}$$

for $m = 0,1,2,3,4$ and

$$N(m,7,7n+5) = \frac{p(7n+5)}{7}$$

for $m = 0,1,2,3,4,5,6$.

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- Dyson conjectured about the existence of a similar statistic $\mathsf{crank}(\lambda)$, with the same properties:
- Denote $M(m, t, n) = \#\{\lambda \text{ partition of } n | \mathsf{crank}(\lambda) \equiv m \pmod{t}\}$. Then the following should hold:

$$M(m, t, n) = M(t - m, t, n)$$

$$M(m, t, tn + r) = \frac{p(tn + r)}{t}$$

for $m \in \{0, \ldots, t - 1\}$ where $(t, r) \in \{(5, 4), (7, 5), (11, 6)\}$.

Dyson’s conjectures resolved

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- 1988: Andrews and Garvan find cranks.
- 1990: Garvan, Kim and Stanton give a single strategy to find cranks for $5n + 4$, $7n + 5$ and $11n + 6$, along with explicit bijections between the crank classes. Our aim is to explain and clarify their work.

Hooks and t-cores

Hooks

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ its conjugate.

1. The (i, j)-*cell of* λ is the cell in row i and column j of the Young-diagram of λ.

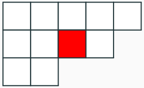

Figure 3: (2,3)-cell of the partition $\lambda = (5, 4, 2)$.

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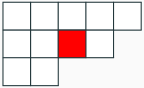

Figure 3: (2,3)-cell of the partition $\lambda = (5, 4, 2)$.

2. The (i,j)-*hook of* λ is the subset consisting of the (i,r)- and (s,j)-cells of λ with $r \geq j$ and $s \geq i$. The (i,j)-hook of λ is denoted H^{λ}_{ij}.

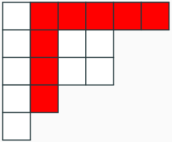

Figure 4: (1,2)-hook of the partition $\lambda = (6, 4, 4, 2, 1)$.

Hooks

3. The number $h_{ij}^{\lambda} = \lambda_i - i + \lambda'_j - j + 1$ of cells in H_{ij}^{λ} is called the *length* of H_{ij}^{λ}.

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5. The set of (r, s)-cells on the rim of λ such that $i \leq r \leq \lambda_i$ and $j \leq s \leq \lambda'_j$ is denoted R^λ_{ij} and called *the associated part of the rim* or the *rim-(i, j)-hook* of λ.

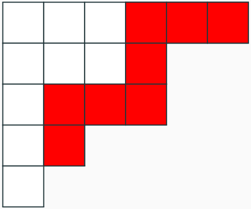

Figure 5: Rim-(1,2)-hook of the partition $\lambda = (6, 4, 4, 2, 1)$.

t-cores

- Pick a hook of λ and remove the associated part of the rim. The resulting diagram is a new Young-diagram. Clearly, the number of cells in R_{ij}^{λ} is the same as in H_{ij}^{λ}.

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- **Definition.** The (i, j)-hook of a partition λ is called a *t-hook* if $h_{ij}^{\lambda} = t$. The associated part of the rim is called a *rim-t-hook*. The partition λ is said to be a *t-core* if it has no hooks of length divisible by t, or equivalently no rim hooks of length divisible by t.

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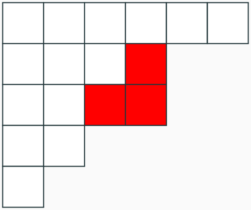
- **Theorem.** Pick a number t and a partition λ. By subsequent removal of rim-t-hooks from λ, one eventually obtains a t-core partition $\tilde{\lambda}$, independent of the sequence of removals. This unique partition $\tilde{\lambda}$ is called the t-core of λ. $\square$

t-cores

We find the 3-core of $\lambda = (6, 4, 4, 2, 1)$: $\tilde{\lambda} = (3, 1, 1)$.



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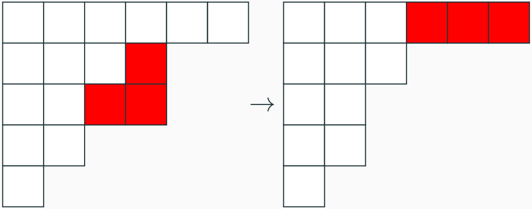

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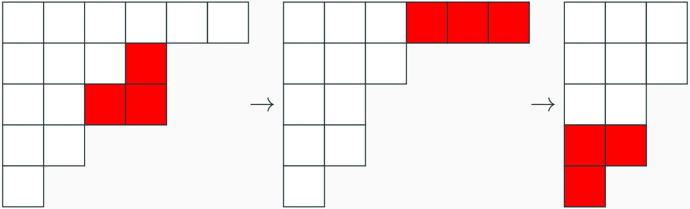

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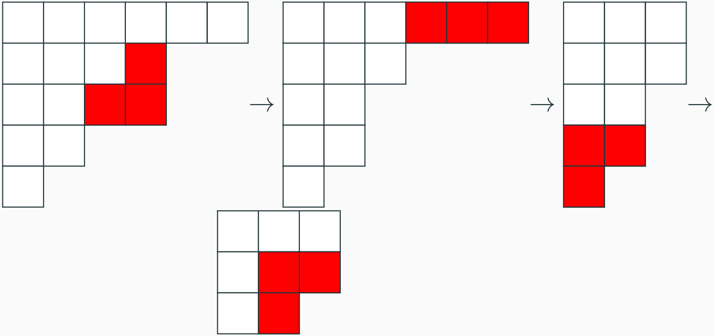

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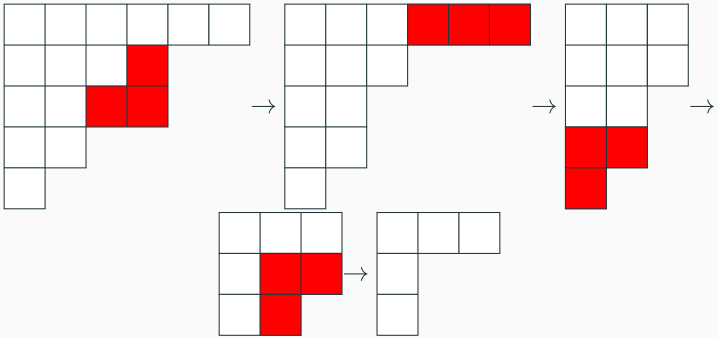

Two bijections

Existence of bijections

Theorem. Let P denote the set of partitions, $P_{t-\mathrm{core}}$ the set of t-cores. There are bijections

$$\phi_1 : P \to P_{t-\mathrm{core}} \times P \times \cdots \times P, \quad \lambda \mapsto (\tilde{\lambda}, \lambda_0, \ldots, \lambda_{t-1}),$$

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where

$$|\tilde{\lambda}| = t \left\|\vec{n}\right\|^2 /2 + \vec{b} \cdot \vec{n}, \quad \vec{b} = (0, 1, \ldots, t-1).$$

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A cell of the extended t-residue diagram is called *exposed* if it is at the (right) end of a row. For $r \in \mathbb{Z}$ we define *region r* of the extended t-residue diagram to be the set of cells such that

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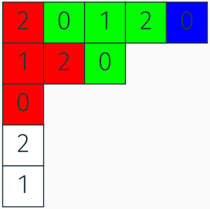
For $i = 0, \ldots t-1$ we define n_i as the maximum number of a region containing an exposed cell labelled i. This number is well defined since column 0 contains infinitely many exposed cells.

ϕ_2 example

Consider $\tilde{\lambda} = (4, 2)$. Then $n_0 = 2$, $n_1 = -1$ and $n_2 = -1$ (see Figure 6, where region -1 of the extended 3-residue diagram is coloured white, region 0 in red, region 1 in green and region 2 in blue).



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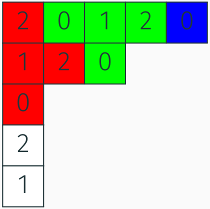

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Remark. If $\phi_2(\lambda) = (n_0, \ldots, n_{t-1})$, then $\phi_2(\lambda') = (-n_{t-1}, \ldots, -n_0)$.

Remark about ϕ_1

The $\tilde{\lambda}$ in the formula

$$\phi_1(\lambda) = (\tilde{\lambda}, \lambda_0, \ldots, \lambda_{t-1})$$

is in fact the t-core of λ. How the λ_i's are defined is not important for now.

Since hook lengths are preserved by conjugation, we note that $\widetilde{(\lambda')} = (\tilde{\lambda})'$.

Construction of cranks

Basis

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- Define

$$\begin{aligned}
\vec{v}_0 &:= \phi_2((5)) = (0,0,0,0,1,0,-1) \\
\vec{v}_1 &:= \phi_2((4,1)) = (0,0,0,1,0,-1,0) \\
\vec{v}_2 &:= \phi_2((3,2)) = (1,0,1,0,0,-1,-1) \\
\vec{v}_3 &:= \phi_2((3,1,1)) = (0,0,1,0,-1,0,0) \\
\vec{v}_4 &:= \phi_2((2,2,1)) = (1,1,0,0,-1,0,-1) \\
\vec{v}_5 &:= \phi_2((2,1,1,1)) = (0,1,0,-1,0,0,0) \\
\vec{v}_6 &:= \phi_2((1,1,1,1,1)) = (1,0,-1,0,0,0,0)
\end{aligned}$$

Quadratic form

Lemma. A vector $\vec{n} \in \mathbb{Z}^7$ satisfies $\vec{n} \cdot \vec{1} = 0$ and $\vec{b} \cdot \vec{n} \equiv 5 \bmod 7$ if, and only if, $\vec{n} = \alpha_0 \vec{v}_0 + \cdots + \alpha_6 \vec{v}_6$ for some $\vec{\alpha} \in \mathbb{Z}^7$ such that $\vec{\alpha} \cdot \vec{1} = 1$. Further, such an $\vec{\alpha}$ is unique. $\square$

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We have

$$\vec{n} = \begin{pmatrix} n_0 \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = \begin{pmatrix} \alpha_2 + \alpha_4 + \alpha_6 \\ \alpha_4 + \alpha_5 \\ \alpha_2 + \alpha_3 - \alpha_6 \\ \alpha_1 - \alpha_5 \\ \alpha_0 - \alpha_3 - \alpha_4 \\ -\alpha_1 - \alpha_2 \\ -\alpha_0 - \alpha_2 - \alpha_4 \end{pmatrix}$$

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and thus

$$\frac{7}{2} \|\vec{n}\|^2 + \vec{b} \cdot \vec{n} = 7 \|\vec{\alpha}\|^2 - 7(\alpha_0\alpha_3 + \alpha_3\alpha_6 + \alpha_6\alpha_2 + \alpha_2\alpha_5 + \alpha_5\alpha_1 + \alpha_1\alpha_4 + \alpha_4\alpha_0) - 2$$

(this number is $|\tilde{\lambda}|$)

Proof of Ramanujan's congruence

By a change of coordinates, one obtains

$$\frac{7}{2}\left\|\vec{n}\right\|^2 + \vec{b}\cdot\vec{n} = 7Q(\vec{\alpha}) - 2$$

where

$$Q(\vec{\alpha}) = \left\|\vec{\alpha}\right\|^2 - \sum_{i=0}^{6}\alpha_i\alpha_{i+1}$$

where the indices in the sum are taken mod 7.

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The quadratic form Q is invariant under the 7-cycle $(0\ 1\ 2\ 3\ 4\ 5\ 6)$ (in fact, the automorphism group is isomorphic to D_7). It has no fixed point $\vec{\alpha}$ (since $\vec{\alpha}\cdot\vec{1} = 1$), thus each of its orbits is of cardinality 7, proving Ramanujan's congruence mod 7.

Crank for $7n + 5$

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A statistic which increases by 1 mod 7 on each cyclic permutation of the coordinates of $\vec{\alpha}$ is

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This statistic satisfies the second condition of being a crank, i.e.

$$M(m, 7, 7n + 5) = \frac{p(7n + 5)}{7}$$

but not the first.

Crank for $7n+5$

Theorem. A crank for partitions λ of $7n+5$ is given by the following algorithm.

(1) Find the 7-core $\tilde{\lambda}$ of λ by the bijection ϕ_1.

(2) Find $\phi_2(\tilde{\lambda}) = \vec{n}$.

(3) Let $\mathrm{crank}(\lambda) = 4n_0 + 2n_1 + n_2 + n_4 + 2n_5 + 4n_6 \bmod 7$.

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In fact, $\mathsf{crank}(\lambda) = 4 + 3\vec{b} \cdot \vec{\alpha}$. Since $x \mapsto 4 + 3x$ is bijective on $\mathbb{Z}/7\mathbb{Z}$, and $\mathsf{crank}(\lambda) = -\mathsf{crank}(\lambda')$, we conclude that $\mathsf{crank}(\lambda)$ satisfies both the conditions imposed by Dyson on his crank.

Thanks for listening!