Solid mathematics

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Introduction

These notes are based on my talk on solid mathematics given in the course *Topics in Algebraic Topology* at the University of Copenhagen in January 2022. The goal is to establish the theory of solid spectra, an analogue in condensed spectra of the theory of solid abelian groups, developed by Clausen and Scholze (see [Sch19b, Lectures V and VI] for an account on solid abelian groups).

We fix an uncountable strong limit cardinal κ. Throughout the text, all profinite sets considered will be κ-small, and what we call a condensed object will actually be κ-condensed. This will allow us to not care about any set-theoretic issues and our ∞-categories of condensed objects will be presentable, allowing us to use results from [Lur09, Section 5.5.4]. We refer to [Sch19b, Appendix to Lecture II] for a discussion about set-theoretic issues and the official definition of a condensed object.

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1 Reminders about condensed spectra

We recall a few facts about condensed spectra.

(1.1) There is an embedding of spectra into condensed spectra (defined as left adjoint to the global sections functor $X \mapsto X(*)$), and the condensed spectra in the essential image of this embedding are called *discrete* condensed spectra. A condensed spectrum X is discrete if and only if for all profinite sets $T = \varprojlim_i T_i$, the map

$$\varinjlim_i X(T_i) \to X(T)$$

is an equivalence. This characterisation also holds for condensed sets, abelian groups and anima.

(1.2) The derived ∞-category of condensed abelian groups, $\mathcal{D}(\mathrm{CondAb})$ is equivalent, via the canonical map

$$\mathcal{D}(\mathrm{Cond\,Ab}) \to \mathrm{Cond}(\mathcal{D}(\mathrm{Ab})),$$

to the ∞-category of condensed objects in the derived ∞-category of abelian groups. They are also equivalent to the ∞-category of $\mathbb{Z}$-module spectra in condensed spectra. In particular, for such spectra X, Y,

$$\underline{\mathrm{map}}_{\mathbb{Z}}(X, Y) \simeq R\underline{\mathrm{Hom}}(X, Y).$$

The underline denotes the internal mapping spectrum which exists in every symmetric monoidal ∞-category whose tensor product commutes with colimits in each variable, and is defined such that $\underline{\mathrm{map}}(X, -)$ is right adjoint to $- \otimes X$ for all X.

(1.3) Condensed cohomology of compact Hausdorff spaces with coefficients in a discrete abelian group agrees with sheaf cohomology. In particular, condensed cohomology of a profinite set T is $C(T, \mathbb{Z})$ concentrated in degree zero.

(1.4) The t-structure on condensed spectra is defined objectwise on extremally disconnected sets. More precisely, $\pi_n X$ for a condensed spectrum X is the condensed abelian group whose value at an extremally disconnected set T is $\pi_n(X(T))$. Since all limits and colimits in condensed spectra are computed objectwise on extremally disconnected sets, this implies that homotopy groups of condensed spectra commute with arbitrary products and filtered colimits (together, these also imply the analogous statement for arbitrary direct sums, by writing them as filtered colimits of finite direct sums, which agree with finite products in this setting). We can equivalently phrase the statement for products as follows: an arbitrary product of connective condensed spectra is connective. All this can be proved using the same type of arguments as found in [Asg21, (2.2.3)–(2.2.5)] when proving that condensed abelian groups satisfy the same AB axioms as abelian groups.

2 Solid abelian groups

We recall the main results of the theory of solid abelian groups, see [Sch19b, Lectures V and VI] for a detailed account. The results we state without proof are proved there. See also [Sch19a, Lecture II] for another discussion including more motivation behind the theory.

(2.1) For a profinite set $T = \varprojlim_i T_i$, we let

$$\mathbb{Z}[T]^{\blacksquare} := \varprojlim_i \mathbb{Z}[T_i].$$

It comes equipped with a canonical natural map $\mathbb{Z}[T] \to \mathbb{Z}[T]^{\blacksquare}$.

(2.2) Definition. A condensed abelian group M is *solid* if the map

$$\operatorname{Hom}\left(\mathbb{Z}[T]^{\blacksquare}, M\right) \to \operatorname{Hom}\left(\mathbb{Z}[T], M\right), \tag{1}$$

induced from the natural map in (2.1), is an isomorphism. An object C of $\mathcal{D}(\mathrm{CondAb})$ is *solid* if the corresponding map

$$R\operatorname{Hom}\left(\mathbb{Z}[T]^{\blacksquare}, C\right) \to R\operatorname{Hom}\left(\mathbb{Z}[T], C\right) \tag{2}$$

is an equivalence.

(2.3) Remark. It follows from the general theory that

(1) A condensed abelian group M is solid if and only if the object $M[0] \in \mathcal{D}(\mathrm{CondAb})$ is solid.

(2) An object C of $\mathcal{D}(\mathrm{CondAb})$ is solid if and only if its condensed cohomology groups $H^i(C)$ are solid abelian groups for all i.

(3) An object is solid if and only if the internal versions of (1) and (2) hold (i.e. with Hom replaced by $\underline{\operatorname{Hom}}$ and $R\operatorname{Hom}$ replaced by $R\underline{\operatorname{Hom}}$).

(2.4) For $T = \varprojlim_i T_i$ a profinite set, we consider the condensed abelian group $C(T,\mathbb{Z}) = \underline{\operatorname{Hom}}\left(\mathbb{Z}[T], \mathbb{Z}\right)$. This is in fact a discrete condensed abelian group (see (2.5)).

(2.5) Lemma. For every profinite set T, there is a κ-small set I and a non-canonical isomorphism of condensed abelian groups

$$C(T,\mathbb{Z}) \simeq \bigoplus_{i\in I} \mathbb{Z}.$$

Proof. See [Sch19b, Theorem 5.4] for a proof of the equivalence

$$C(T,\mathbb{Z})(*) \simeq \left(\bigoplus_{i\in I} \mathbb{Z}\right)(*).$$

To see that it extends to an isomorphism of the condensed abelian groups, we use the characterisation of discrete condensed objects in (1.1) to show that $C(T, \mathbb{Z})$ is in fact discrete. Indeed, its S-valued points for a profinite set $S = \varprojlim_j S_j$ are

$$\begin{aligned} C(T,\mathbb{Z})(S) &\simeq C(T \times S, \mathbb{Z})(*) \\ &\simeq \varinjlim_{i,j} C(T_i \times S_j, \mathbb{Z})(*) \\ &\simeq \varinjlim_{j} \varinjlim_{i} C(T_i \times S_j, \mathbb{Z})(*) \\ &\simeq \varinjlim_{j} C(\varprojlim_{i} T_i \times S_j, \mathbb{Z})(*) \\ &\simeq \varinjlim_{j} C(T, \mathbb{Z})(S_j), \end{aligned}$$

since $\mathbb{Z}$ is discrete. □

(2.6) Remark. By similar calculations to the ones in the proof of (2.5), we can show that for every discrete condensed abelian group A and profinite set T, the canonical map

$$\varinjlim_{i} C(T_i, A) \to C(T, A)$$

is an equivalence. This extends to discrete condensed spectra and will be used in that setting in the section 3.

(2.7) Corollary. For every profinite set T, there is a κ-small set I and a non-canonical isomorphism of condensed abelian groups

$$\mathbb{Z}[T]^{\blacksquare} \simeq \prod_{i \in I} \mathbb{Z}.$$

Proof. We have an isomorphism $\bigoplus_{i \in I} \mathbb{Z} \simeq C(T, \mathbb{Z})$. Using the fact that $C(T, \mathbb{Z})$ is discrete, we see that

$$\mathbb{Z}[T]^{\blacksquare} \simeq \underline{\mathrm{Hom}}\,(C(T,\mathbb{Z}), \mathbb{Z}) \simeq \underline{\mathrm{Hom}} \left(\bigoplus_{i \in I} \mathbb{Z}, \mathbb{Z} \right) \simeq \prod_{i \in I} \mathbb{Z}.$$

The first equivalence follows from (2.6), in this case $\varinjlim_i C(T_i, \mathbb{Z}) \simeq C(T, \mathbb{Z})$. The last equivalence is a priori only true on underlying abelian groups, but it follows formally for the condensed abelian groups by showing that they corepresent the same functor on CondAb. □

(2.8) Theorem. For every (κ-small) profinite set T, $\mathbb{Z}[T]^{\blacksquare}$ is solid. □

We denote by Solid Ab the full subcategory of Cond Ab spanned by the solid abelian groups.

(2.9) Theorem. ([Sch19b, Theorem 5.8])

(1) The category of solid abelian groups is an abelian category generated by compact projectives of the form $\prod_{i \in I} \mathbb{Z}$. Further, the fully faithful inclusion $i : \mathrm{Solid\,Ab} \hookrightarrow \mathrm{CondAb}$ preserves all limits, colimits and extensions and has a left adjoint denoted $M \mapsto M^{\blacksquare}$ which is a colimit-preserving extension of $\mathbb{Z}[T] \mapsto \mathbb{Z}[T]^{\blacksquare}$, and as such, unique up to unique isomorphism.

(2) The functor $i : \mathcal{D}(\mathrm{Solid\,Ab}) \to \mathcal{D}(\mathrm{CondAb})$ induced from i above is fully faithful and its essential image is spanned by the solid objects of $\mathcal{D}(\mathrm{CondAb})$. It admits a left adjoint $C \mapsto C^{\blacksquare}$ which is the left derived functor of the solidification functor on condensed abelian groups. Also, it is a colimit-preserving extension of $\mathbb{Z}[T] \mapsto \mathbb{Z}[T]^{\blacksquare}$ and as such, unique up to contractible choice. An object $C \in \mathcal{D}(\mathrm{CondAb})$ is solid if and only if $H^i(C)$ is solid for all i. □

(2.10) Theorem. There is a unique way to endow the category Solid Ab with a symmetric monoidal tensor product $\otimes^{\blacksquare}$, making the functor $M \mapsto M^{\blacksquare}$ symmetric monoidal.

(2.11) Remark. A similar statement to (2.10) holds in the derived setting (see (3.14)).

3 Solid spectra

The theory of solid spectra is developed analogously to that of solid objects in derived condensed abelian groups. Our most important results ((3.4), (3.7)) will be tools to reduce statements to analogues for solid abelian groups. We obtain the main two results (3.13) and (3.14), analogues to (2.9) and (2.10).

(3.1) For a profinite set $T = \varprojlim_i T_i$, we let

$$\mathbb{S}[T]^{\blacksquare} := \varprojlim_i \mathbb{S}[T_i].$$

It comes equipped with a canonical natural map $\mathbb{S}[T] \to \mathbb{S}[T]^{\blacksquare}$.

(3.2) Definition. A condensed spectrum X is *solid* if the map

$$\operatorname{map}(\mathbb{S}[T]^{\blacksquare}, X) \to \operatorname{map}(\mathbb{S}[T], X) \tag{3}$$

induced from the natural map in (3.1), is an equivalence.

(3.3) For a condensed spectrum X, we let $C(T, X)$ denote the condensed mapping spectrum $\underline{\operatorname{map}}(\mathbb{S}[T], X)$. For the discrete condensed spectra $\mathbb{S}$ and $\mathbb{Z}$, we have a unique map of commutative algebras $\mathbb{S} \to \mathbb{Z}$ in $\operatorname{Cond}\operatorname{Sp}^{\otimes}$, yielding a map

$$C(T, \mathbb{S}) \to C(T, \mathbb{Z}).$$

Further, since $C(T, \mathbb{Z})$ is a $\mathbb{Z}$-module in condensed spectra, we have a map

$$C(T, \mathbb{S}) \otimes_{\mathbb{S}} \mathbb{Z} \to C(T, \mathbb{Z}).$$

(3.4) Lemma. The map

$$C(T, \mathbb{S}) \otimes_{\mathbb{S}} \mathbb{Z} \to C(T, \mathbb{Z})$$

from (3.3) is an equivalence.

Proof. First a note on how to interpret the right hand side in the equivalence. For the sake of this argument, we will temporarily denote the Eilenberg-Maclane spectrum associated to an abelian group A by HA and not just A. We want to show that $HC(T, \mathbb{Z}) \simeq C(T, H\mathbb{Z})$, where in the left hand side, $C(T, \mathbb{Z})$ denotes the condensed abelian group, and in the right hand side, $C(T, H\mathbb{Z})$ is the condensed spectrum. Now,

$$\begin{aligned} C(T, H\mathbb{Z}) &\simeq \underline{\operatorname{map}}(\mathbb{S}[T], H\mathbb{Z}) \\ &\simeq \underline{\operatorname{map}}_{H\mathbb{Z}}(\mathbb{S}[T] \otimes_{\mathbb{S}} H\mathbb{Z}, H\mathbb{Z}) \\ &\simeq R\underline{\operatorname{Hom}}(\mathbb{Z}[T], \mathbb{Z}) \\ &\simeq H\underline{\operatorname{Hom}}(\mathbb{Z}[T], \mathbb{Z}) \\ &\simeq HC(T, \mathbb{Z}) \end{aligned}$$

where in the penultimate step, we used the fact that the condensed cohomology of a profinite set is concentrated in degree 0. In other words, the internal mapping spectrum in question lies in the heart of the t-structure on derived condensed abelian groups.

Now for the actual proof of the lemma, we use the fact that $\mathbb{S}$ and $\mathbb{Z}$ are discrete, and by the analogue of (2.6) for spectra, we can write

$$C(T, \mathbb{S}) \simeq \varinjlim_i C(T_i, \mathbb{S}) \simeq \varinjlim_i \bigoplus_{T_i} \mathbb{S},$$

$$C(T,\mathbb{Z}) \simeq \varinjlim_i C(T_i,\mathbb{Z}) \simeq \varinjlim_i \bigoplus_{T_i} \mathbb{Z},$$

and use the fact that the tensor product commutes with colimits in each variable. □

(3.5) Corollary. For every profinite set T, there is a κ-small set I and a non-canonical equivalence of condensed spectra

$$C(T,\mathbb{S}) \simeq \bigoplus_{i\in I} \mathbb{S}.$$

Proof. We need to produce a map

$$\bigoplus_{i\in I} \mathbb{S} \to C(T,\mathbb{S})$$

which after extension of scalars along $\mathbb{S}\to\mathbb{Z}$ becomes the equivalence

$$\bigoplus_{i\in I} \mathbb{Z} \simeq C(T,\mathbb{Z}) \simeq C(T,\mathbb{S}) \otimes_{\mathbb{S}} \mathbb{Z}$$

that we already have. This is enough because the condensed spectra we are working with are in fact discrete, and a map $X \to Y$ of connective spectra is an equivalence if and only if the induced $X \otimes_{\mathbb{S}} \mathbb{Z} \to Y \otimes_{\mathbb{S}} \mathbb{Z}$ is an equivalence. For each $i \in I$ we have a map of $\mathbb{Z}$-modules in condensed spectra

$$p_i : \mathbb{Z} \to C(T,\mathbb{Z})$$

which gives a

$$\tilde{p}_i : \mathbb{S} \to C(T,\mathbb{Z})$$

by restriction of scalars along $\mathbb{S}\to\mathbb{Z}$. We thus have elements

$$\begin{aligned} \tilde{p}_i \in \pi_0 \operatorname{map}(\mathbb{S}, C(T,\mathbb{Z})) &\simeq C(T,\mathbb{Z}) \\ &\simeq C(T,\pi_0\mathbb{S}) \\ &\simeq \pi_0 C(T,\mathbb{S}) \\ &\simeq \pi_0 \operatorname{map}(\mathbb{S}, C(T,\mathbb{S})) \end{aligned}$$

where, when pulling out the π_0, we use a similar argument to the first part of the proof of (3.4). These $\tilde{p}_i$ assemble into the desired map

$$\bigoplus_{i\in I} \mathbb{S} \to C(T,\mathbb{S}).$$

□

(3.6) Corollary. For every profinite set T, there is a κ-small set I and a non-canonical equivalence of condensed spectra

$$\mathbb{S}[T]^{\blacksquare} \simeq \prod_{i\in I} \mathbb{S}.$$

Proof. The proof is identical to the one for abelian groups (2.7). □

We are now ready to prove the theorem which will be our main tool in reducing the proofs of results in the theory of solid spectra to known results about solid abelian groups.

(3.7) Theorem. For every (κ-small) profinite set T,

$$\mathbb{S}[T]^{\blacksquare} \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathbb{Z}[T]^{\blacksquare}$$

(3.8) Remark. Before proving the theorem, we note that it identifies solid objects of the derived category of condensed abelian groups and solid spectra with the structure of a $\mathbb{Z}$-module in condensed spectra: Indeed, if C is a $\mathbb{Z}$-module in condensed spectra, or equivalently an object of $\mathcal{D}(\mathrm{CondAb})$, then by the theorem and restriction of scalars,

$$\begin{aligned} \mathrm{map}\left(\mathbb{S}[T]^{\blacksquare}, C\right) &\simeq \mathrm{map}_{\mathbb{Z}}\left(\mathbb{S}[T]^{\blacksquare} \otimes_{\mathbb{S}} \mathbb{Z}, C\right) \\ &\simeq R\,\mathrm{Hom}\left(\mathbb{Z}[T]^{\blacksquare}, C\right). \end{aligned}$$

Further, the analogous equivalence is true when $\mathbb{S}[T]^{\blacksquare}, \mathbb{Z}[T]^{\blacksquare}$ are replaced by $\mathbb{S}[T], \mathbb{Z}[T]$. We conclude that C is solid as a spectrum if and only if it is solid as an object of $\mathcal{D}(\mathrm{CondAb})$.

In particular, a condensed abelian group A is solid if and only if the Eilenberg-Maclane spectrum A is solid.

To prove Theorem (3.7), we need two lemmas ((3.9) and (3.10)). First, recall that a spectrum is *finite* if it can be written as a finite colimit of shifts of $\mathbb{S}$. The finite spectra are compact objects of Sp.

(3.9) Lemma. For every connective spectrum X with finitely generated homotopy groups, there is a simplicial spectrum $X_\bullet$ with geometric realisation X such that X_n is a finite spectrum for all n.

Proof. We let $Y_0 \to X$ be a finite spectrum mapping to X which is surjective on π_0 (Y_0 can be taken as a finite direct sum of copies of $\mathbb{S}$). We take $C_0 = Y_0$ and let F_0 be the fibre of $C_0 \to X_0$. It's connective because $C_0 \to X$ is surjective on π_0 and shifting the fibre sequence up we get that

$$\tau_{\leq 0}(F_0[1]) \simeq \pi_0(F_0[1]) \simeq \mathrm{coker}\left(\pi_0 C_0 \to \pi_0 X_0\right) \simeq 0,$$

i.e. $F_0[1]$ is 1-connective and thus F_0 is connective. The first equivalence above is because $F_0[1]$ is connective as a colimit of connective spectra, and the second because π_0 commutes with colimits of connective spectra. Also, F_0 has finitely generated π_0, because $\pi_1 X$ and $\pi_0 C_0$ are finitely generated and $\pi_0 F_0$ sits between them in the long exact sequence. Thus, we can again find a finite spectrum Y_1 mapping to F_0 surjectively on π_0 and now we let C_1 be the cofibre of $Y_1 \to C_0$. Then we want to show that $C_1 \to X$ is surjective on π_1. More generally, the inductive step is as follows: we let F_{n-1} be the fibre of $C_{n-1} \to X$, $Y_n \to F_{n-1}$ a map from a finite direct sum of copies of $\mathbb{S}[n-1]$, surjective on π_{n-1}, and C_n the cofibre of $Y_n \to C_{n-1}$. The following diagram consists of parts of the two long exact sequences corresponding to the fibre sequences $F_{n-1} \to C_{n-1} \to X$ and $Y_n \to C_{n-1} \to C_n$:

$$\begin{array}{ccccccc} \pi_n C_{n-1} & \longrightarrow & \pi_n C_n & \longrightarrow & \pi_{n-1} Y_n & \longrightarrow & \pi_{n-1} C_n \\ \| & & \downarrow & & \downarrow & & \| \\ \pi_n C_{n-1} & \longrightarrow & \pi_n X & \longrightarrow & \pi_{n-1} F_{n-1} & \longrightarrow & \pi_{n-1} C_{n-1} \end{array}$$

Since the equality on the left is an epimorphism, the equality on the right a monomorphism and $\pi_{n-1} Y_n \to \pi_{n-1} F_{n-1}$ an epimorphism, the map $\pi_n C_n \to \pi_n X$ is an epimorphism, by a diagram chase (one of the 4-lemmas, which can be used to prove the better-known 5-lemma).

Inductively we get a sequence $C_0 \to C_1 \to C_2 \to \cdots$, such that $C_n \to X$ is surjective on π_n and C_n is a finite spectrum for all n. We will show that its colimit is X, and then by the ∞-categorical Dold-Kan correspondence (see [Lur17, Theorem 1.2.4.1]) we get the desired simplicial finite spectrum.

To see that $\varinjlim_n C_n \simeq X$, it is enough to prove that the map $\varinjlim_n C_n \to X$ induces isomorphisms on homotopy groups. These commute with the filtered colimit so it suffices to show that

$$\varinjlim_n \pi_i(C_n) \to \pi_i(X)$$

is an isomorphism for all i. In fact, we show that

$$\pi_k(C_{n+1}) \to \pi_k(X)$$

is an isomorphism for all $k \leq n$. We will simultaneously prove the above isomorphism and the fact that F_n is n-connective by induction on n. For the base case, we have already proved that F_0 is connective. We also note that $\pi_{-1}C_0 \to \pi_{-1}X$ is an isomorphism as both are 0 (if we want to start the induction one step higher, the isomorphism $\pi_0 C_1 \simeq \pi_0 X$ can be proved in exactly the same way as the inductive step). Now suppose $\pi_{k-1}C_n \to \pi_{k-1}X$ is an isomorphism for all $k \leq n$. Using this and a similar argument to the one for connectivity of F_0, we see that F_n is n-connective. Now if we consider the diagram

$$\begin{array}{ccccccccc}
\pi_k Y_{n+1} & \longrightarrow & \pi_k C_n & \longrightarrow & \pi_k C_{n+1} & \longrightarrow & \pi_{k-1} Y_{n+1} & \longrightarrow & \pi_{k-1} C_n \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| \\
\pi_k F_n & \longrightarrow & \pi_k C_n & \longrightarrow & \pi_k X & \longrightarrow & \pi_{k-1} F_n & \longrightarrow & \pi_{k-1} C_n
\end{array}$$

obtained from the long exact sequences corresponding to the fibre sequences $Y_{n+1} \to C_n \to C_{n+1}$ and $F_n \to C_n \to X$, we see that the vertical morphism furthest to the left is an epimorphism (it is actually an isomorphism when $k < n$, but not necessarily when $k = n$) while the other three vertical morphisms excluding the middle one are isomorphisms (in $\pi_{k-1}Y_{n+1} \to \pi_{k-1}F_n$, we have now established that both source and target are 0). The 5-lemma implies that the vertical morphism in the middle is an isomorphism, as desired. ☐

(3.10) Lemma. If I is an ∞-category and $(X_{i,\bullet})_i$ a diagram of simplicial connective condensed spectra such that for all n, the limit $\lim_I X_{i,n}$ is connective[1], then the limit commutes with geometric realisation. More precisely, if we let $\lim_I X_{i,\bullet}$ denote the simplicial spectrum obtained by taking the degreewise limit, then

$$\left|\lim_I X_{i,\bullet}\right| \simeq \lim_I |X_{i,\bullet}| \, .$$

Proof. Since everything is connective, we can apply the ∞-categorical Dold-Kan correspondence, so it is enough to prove the equivalence

$$\varinjlim_n \left|\mathrm{sk}_n \lim_I X_{i,\bullet}\right| \simeq \lim_I \varinjlim_n |\mathrm{sk}_n X_{i,\bullet}|$$

In a range of degrees, we can drop the $\varinjlim_n$. By [Lur09, Corollary 5.1.2.3] and its dual, limits and colimits in a functor category are computed objectwise. Thus the category $\mathcal{C}^I$ is stable for every stable ∞-category $\mathcal{C}$. Therefore by [Lur17, Proposition 1.1.4.1], $\lim_I$ commutes with finite colimits (since it commutes with finite limits), in particular it commutes with the geometric realisations of skeleta above, as desired. ☐

Proof of Theorem (3.7). Apply Lemma (3.9) to the connective spectrum $\mathbb{Z}$, commute the tensor product with colimits and finally commute the product with finite colimits (possible because of stability) and then, using Lemma (3.10), commute it with the geometric realisation. ☐

(3.11) Corollary. If a condensed spectrum X has solid homotopy groups, then X is a solid spectrum.

Proof. First suppose X is bounded below. It is easy to see that solid spectra are closed under limits, extensions and shifts. Writing $X \simeq \varprojlim_n \tau_{\leq n} X$ we then see that it suffices to show that $\tau_{\leq n}X$ is solid for all n. We can prove this by induction using the fibre sequences

$$(\pi_n X)[n] \to \tau_{\leq n} X \to \tau_{\leq n-1} X$$

where the inductive step follows from the assumption that all homotopy groups are solid (as spectra by Remark (3.8)) and the fact that solid spectra are closed under extensions and shifts. The fact that X is bounded below allows us to get the induction started.

[1] This holds for instance if it is a product, or a limit of Postnikov truncations of a connective spectrum. In fact, the former is the only case to which we will apply this result.

To extend to the general case, we note that for every $d \in \mathbb{Z}$,

$$\pi_d \operatorname{map}(\mathbb{S}[T]^{\blacksquare}, X) \to \pi_d \operatorname{map}(\mathbb{S}[T], X)$$

only depends on $\tau_{\geq -d}X$. Indeed, for every Y,

$$\pi_d \operatorname{map}(Y, X) \simeq \pi_0 \operatorname{map}(Y, X[d]) \simeq \operatorname{Map}(Y, X[d])$$

and if Y is connective, there are no maps in negative degree. Since $\mathbb{S}[T]^{\blacksquare}$ and $\mathbb{S}[T]$ are connective, we are done. □

(3.12) Theorem. For every profinite set T, $\mathbb{S}[T]^{\blacksquare}$ is solid.

Proof. For formal reasons (mapping into a product), it suffices to check that the spectrum $\mathbb{S}$ is solid. By (3.11), it suffices to show that $\pi_n\mathbb{S}$ is a solid abelian group for all n. This is clear because these are all built from $\mathbb{Z}$ (which is solid, see [Sch19b, Proposition 5.7]) by colimits, limits and extensions, and solid abelian groups are closed under these by (2.9). □

(3.13) Theorem. The category of solid spectra is a stable ∞-category generated under shifts and colimits by compact projectives of the form $\prod_{i\in I} \mathbb{S}$. Further, the fully faithful inclusion $i : \mathrm{Solid\,Sp} \hookrightarrow \mathrm{Cond\,Sp}$ preserves all limits, colimits and extensions and has a left adjoint denoted $X \mapsto X^{\blacksquare}$ which is a colimt-preserving extension of $\mathbb{S}[T] \mapsto \mathbb{S}[T]^{\blacksquare}$, and as such, unique up to contractible choice. Furthermore, a condensed spectrum X is solid if and only if all of its homotopy groups $\pi_n X$ are solid abelian groups.

Proof. We have already noted that solid spectra are closed under limits and extensions. Closure under colimits follows from the fact that Solid Sp is generated under colimits by the objects of the form $\mathbb{S}[T]^{\blacksquare}[n]$ for T profinite and $n \in \mathbb{Z}$, which is the main body of the proof that now follows.

We apply [Lur09, Proposition 5.5.4.15] to the collection S of all morphisms $\mathbb{S}[T]^{\blacksquare}[n] \to 0$, where T is profinite and $n \in \mathbb{Z}$. It implies that we have a functor $L : \mathrm{Cond\,Sp} \to \mathrm{Cond\,Sp}$ and for each condensed spectrum X a map $X \to L(X)$ belonging to the strongly saturated class of maps generated by S, in particular, belongs to the full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cond\,Sp})$ generated under colimits by S (see [Lur09, Definition 5.5.4.5]). This implies that the fibre $F(X)$ of $X \to L(X)$ belongs to the full subcategory of Cond Sp generated under colimits by $\mathbb{S}[T]^{\blacksquare}[n]$, T profinite, $n \in \mathbb{Z}$.

Furthermore, $L(X)$ is S-local, meaning (see [Lur09, Definition 5.5.4.1]) that for every profinite set T and every integer n, the map

$$0 \simeq \operatorname{Map}(0, L(X)) \to \operatorname{Map}(\mathbb{S}[T]^{\blacksquare}[n], L(X)) \simeq \pi_{-n} \operatorname{map}(\mathbb{S}[T]^{\blacksquare}, L(X))$$

is an equivalence, i.e. that the spectrum of maps $\mathbb{S}[T]^{\blacksquare} \to L(X)$ is zero.

We conclude that the map

$$\operatorname{map}(\mathbb{S}[T]^{\blacksquare}, F(X)) \to \operatorname{map}(\mathbb{S}[T]^{\blacksquare}, X)$$

is an equivalence.

We want to show that $F(X)$ is solid. We show that its homotopy groups are solid, by showing that the full subcategory $\mathrm{Solid\,Sp}'$ of Cond Sp, spanned by the condensed spectra whose homotopy groups are solid, is closed under colimits. This indeed proves that $F(X)$ has solid homotopy groups, since it is generated under colimits by the objects $\mathbb{S}[T]^{\blacksquare}[n]$, which have solid homotopy groups.

Now for the proof that $\mathrm{Solid\,Sp}'$ is stable under colimits, it suffices to show that it is stable under cofibres and direct sums. Since homotopy groups commute with direct sums, the latter is clear. If $Y \to Z \to W$ is a fibre sequence of condensed spectra such that Y and Z have solid homotopy groups, then by the long exact sequence in homotopy and the fact that solid abelian groups are closed under extensions, the homotopy groups of W are solid.

We have established that all objects in the full subcategory of Cond Sp generated under colimits by the objects $\mathbb{S}[T]^{\blacksquare}[n]$ are solid. Conversely, if X is solid, the map $F(X) \to X$ must be an equivalence, because for every profinite set T, we have

$$\begin{aligned} F(X)(T) &\simeq \operatorname{map}(\mathbb{S}[T], F(X)) \\ &\simeq \operatorname{map}(\mathbb{S}[T]^{\blacksquare}, F(X)) \\ &\simeq \operatorname{map}(\mathbb{S}[T]^{\blacksquare}, X) \\ &\simeq \operatorname{map}(\mathbb{S}[T], X) \\ &\simeq X(T). \end{aligned}$$

Therefore, X is generated under colimits and shifts by the objects $\mathbb{S}[T]^{\blacksquare}$.

By the mapping property, the objects $\mathbb{S}[T]^{\blacksquare}$, for T extremally disconnected, are compact projectives. Since a retract of a product of copies of $\mathbb{S}$ is again such a product (see argument in [Sch19b, Proof of Corollary 6.1]), we have the desired statement about the objects of the form $\prod_{i \in I} \mathbb{S}$ being compact projective generators.

Since the generators have solid homotopy groups, we also conclude that a solid spectrum has solid homotopy groups.

As for the solidification functor $X \mapsto X^{\blacksquare}$ left adjoint to the inclusion, its existence is clear by closure under limits. Let's denote it by G temporarily and show that $G(\mathbb{S}[T]) \simeq \mathbb{S}[T]^{\blacksquare}$ for all profinite sets T. For every solid spectrum X, we have

$$\operatorname{map}(G(\mathbb{S}[T]), X) \simeq \operatorname{map}(\mathbb{S}[T], X) \simeq \operatorname{map}(\mathbb{S}[T]^{\blacksquare}, X)$$

so $\mathbb{S}[T]^{\blacksquare}$ and $G(\mathbb{S}[T])$ represent the same functor on Solid Sp and are hence equivalent. □

(3.14) Solid tensor product. We finish this note by promoting the adjunction

$$\operatorname{Cond}\operatorname{Sp} \underset{i}{\overset{(-)^{\blacksquare}}{\rightleftarrows}} \operatorname{Solid}\operatorname{Sp}$$

to a symmetric monoidal adjunction

$$\operatorname{Cond}\operatorname{Sp}^{\otimes} \underset{i^{\otimes}}{\overset{(-)^{\blacksquare,\otimes}}{\rightleftarrows}} \operatorname{Solid}\operatorname{Sp}^{\otimes},$$

in a unique (up to contractible choice) way. We will denote the tensor product obtained on Solid Sp by $\otimes^{\blacksquare}$ and in fact we will have

$$X \otimes^{\blacksquare} Y \simeq (X \otimes Y)^{\blacksquare}.$$

Let W be the collection of morphisms $X \to Y$ in Cond Sp such that the induced $X^{\blacksquare} \to Y^{\blacksquare}$ is an equivalence. Then we see that

$$\operatorname{Solid}\operatorname{Sp} \hookrightarrow \operatorname{Cond}\operatorname{Sp} \to \operatorname{Cond}\operatorname{Sp}[W^{-1}]$$

is an equivalence by [Lur17, Example 1.3.4.3]. Thus by [Lur17, Proposition 4.1.7.4] it suffices to show that if $X \to Y$ becomes an equivalence after solidification, and Z is a condensed spectrum, then the induced maps

$$(X \otimes Z)^{\blacksquare} \to (Y \otimes Z)^{\blacksquare}, \qquad (Z \otimes X)^{\blacksquare} \to (Z \otimes Y)^{\blacksquare}$$

are equivalences. We only consider the former, as the latter is completely analogous. Since we know that

$$(X^{\blacksquare} \otimes Z)^{\blacksquare} \to (Y^{\blacksquare} \otimes Z)^{\blacksquare}$$

is an equivalence, it suffices to show that for every condensed spectrum X, the map

$$(X \otimes Z)^{\blacksquare} \to (X^{\blacksquare} \otimes Z)^{\blacksquare},$$

induced by the unit of the adjunction, is an equivalence. Thanks to Lemma (3.15), to prove that the solidification functor is symmetric monoidal, one can copy [Sch19b, Proof of Theorem 6.2] replacing $\mathbb{Z}$ by $\mathbb{S}$ and Hom by map. The uniqueness statement about $\otimes^{\blacksquare}$ follows from [Lur17, Remark 4.1.7.5].

(3.15) Lemma. A condensed spectrum X is solid if and only if for every profinite set T, the map

$$\underline{\mathrm{map}}\,(\mathbb{S}[T]^{\blacksquare}, X) \to \underline{\mathrm{map}}\,(\mathbb{S}[T], X)$$

is an equivalence.

Proof. Evaluating at the point, one direction is clear. For the other one, we reduce to the case of abelian groups by using the characterisation that X is solid if and only if $\pi_n X$ is a solid abelian group for all n. □

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