Discrete condensed objects

Dagur Ásgeirsson

In this note we give a useful characterisation of discrete condensed anima. As a corollary, we deduce the same characterisation for discrete condensed spectra. I would like to thank Dustin Clausen for outlining the argument, and Lars Hesselholt for his thorough reading and helpful comments on drafts of this document.

(1) Let $\mathcal{C}$ be an ∞-category. Consider the global sections functor $\mathrm{Cond}(\mathcal{C}) \to \mathcal{C}$, $X \mapsto X(*)$. It admits a fully faithful left adjoint, which we denote by $X \mapsto \underline{X}$ throughout this note, but in general, one wants to drop the underline and just identify the object X with its image in condensed objects.

(2) Definition. Let $\mathcal{C}$ be an ∞-category. An object $X \in \mathrm{Cond}(\mathcal{C})$ is *discrete* if it belongs to the essential image of the left adjoint to the global sections functor described in (1).

(3) Lemma. The presheaf of ∞-categories on the site of profinite sets, $T \mapsto \mathrm{Sh}(T)$, is a sheaf.

The proof of this lemma relies on two deep theorems which we now state.

(4) Theorem. (Barr-Beck-Lurie, [Lur17, Corollary 4.7.5.3]). Let $\mathcal{C}^\bullet$ be an augmented cosimplicial ∞-category and set $\mathcal{C} = \mathcal{C}^{-1}$. Let $G : \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that

(1) The ∞-category $\mathcal{C}$ admits geometric realisations of G-split simplicial objects and those geometric realisations are preserved by G.

(2) For every morphism $\alpha : [m] \to [n]$ in Δ_+, the induced diagram

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \xrightarrow[d^0]{} & \mathcal{C}^{n+1} \end{array}$$

is left adjointable.

Then the canonical map $\theta : \mathcal{C} \to \varprojlim_{[n]\in\Delta} \mathcal{C}^n$ admits a fully faithful left adjoint. If G is conservative, then θ is an equivalence. □

(5) Theorem. (The proper base change theorem, [Lur09, Corollary 7.3.1.18]). If

$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow[q]{} & Y \end{array}$$

is a pullback diagram of locally compact Hausdorff spaces and if p is a proper map, then the associated diagram

$$\begin{array}{ccc} \mathrm{Sh}(X') & \xrightarrow{q'_*} & \mathrm{Sh}(X) \\ {\scriptstyle p'_*}\big\downarrow & & \big\downarrow{\scriptstyle p_*} \\ \mathrm{Sh}(Y') & \xrightarrow[q_*]{} & \mathrm{Sh}(Y) \end{array}$$

is left adjointable. □

(6) Remark. In the case we are interested in, the spaces X, Y, X', Y' will be compact Hausdorff spaces, and all continuous maps between such spaces are proper. Also, we will consider the opposites of the sheaf categories, turning the conclusion of the theorem into the fact that the diagram

$$\begin{array}{ccc} \mathrm{Sh}(Y)^{\mathrm{op}} & \xrightarrow{p^{*,\mathrm{op}}} & \mathrm{Sh}(X)^{\mathrm{op}} \\ {\scriptstyle q^{*,\mathrm{op}}}\big\downarrow & & \big\downarrow{\scriptstyle q'^{*,\mathrm{op}}} \\ \mathrm{Sh}(Y')^{\mathrm{op}} & \xrightarrow[p'^{*,\mathrm{op}}]{} & \mathrm{Sh}(X')^{\mathrm{op}} \end{array}$$

is left adjointable, because applying the op-endofunctor on the ∞-category of ∞-categories takes the left adjoint in an adjoint pair to a right adjoint and vice versa, and changes direction of natural transformations.

Proof of Lemma (3). We need to show that for any surjection $f : S \to T$ of profinite sets, the induced functor

$$\mathrm{Sh}(T) \to \varprojlim_{[n]\in\Delta} \mathrm{Sh}\left(S^{\times_T [n]}\right)$$

is an equivalence of ∞-categories. It suffices to show that the functor induced on opposite categories is an equivalence.

We apply Theorem (4), i.e. [Lur17, Corollary 4.7.5.3], to the augmented cosimplicial category $\mathcal{C}^\bullet$, where $\mathcal{C}^{-1} = \mathrm{Sh}(T)^{\mathrm{op}}$, and for $n \geq 0$, $\mathcal{C}^n = \mathrm{Sh}\left(S^{\times_T [n]}\right)^{\mathrm{op}}$. The functor G will be $f^{*,\mathrm{op}}$. There is no mathematical reason to take opposite categories, it is only to conform with the statement in [Lur17].

Condition (2) of Theorem (4) follows from the proper base change theorem (see (5) above or [Lur09, Corollary 7.3.1.18]).

Since f is surjective, the functor f^* (and thus $G = f^{*,\mathrm{op}}$) is conservative.

It remains to verify condition (1) of [Lur17, Corollary 4.7.5.3]. It suffices to show that $f^* : \mathrm{Sh}(T) \to \mathrm{Sh}(S)$ preserves totalisations of cosimplicial objects. We let $\mathcal{F}^\bullet$ be a cosimplicial object in $\mathrm{Sh}(T)$ and write its totalisation as follows

$$\varprojlim_{\Delta} \mathcal{F}^\bullet \simeq \varprojlim_{n\in\mathbb{N}} \varprojlim_{\Delta_{\leq n}} \mathcal{F}^\bullet|_{\Delta_{\leq n}}.$$

Since f^* commutes with finite limits, we have

$$\varprojlim_{n\in\mathbb{N}} f^* \left(\varprojlim_{\Delta_{\leq n}} \mathcal{F}^\bullet|_{\Delta_{\leq n}} \right) \simeq \varprojlim_{n\in\mathbb{N}} \left(\varprojlim_{\Delta_{\leq n}} f^*\mathcal{F}^\bullet|_{\Delta_{\leq n}} \right)$$

so to conclude it suffices to note that f^* commutes with homotopy groups, which follows from [Lur09, Remark 6.5.1.4]. □

(7) Theorem. Let Y be a condensed anima. The following are equivalent

(1) Y is discrete.

(2) For every profinite set $T = \varprojlim_i T_i$, the canonical right hand map in

$$\varinjlim_i \prod_{t \in T_i} Y(\{t\}) \leftarrow \varinjlim_i Y(T_i) \to Y(T)$$

is an equivalence. The left hand map is an equivalence by assumption; the fact that Y is a sheaf implies that it preserves finite products, and T_i is a finite coproduct of its singleton subsets.

Proof. We let X be an anima and define a presheaf $\tilde{X}$ of anima on the site of profinite sets, by letting

$$\tilde{X}(T) = \varinjlim_i \prod_{t \in T_i} X$$

for every profinite set $T = \varprojlim_i T_i$. We claim that the canonical map $\tilde{X} \to \underline{X}$ is an equivalence. Granting this, the theorem follows. Indeed, if Y satisfies (1), then there is an anima X and an equivalence $\underline{X} \simeq Y$. In particular $Y(*) \simeq X$, so

$$\varinjlim_i \prod_{t \in T_i} Y(*) \simeq \tilde{X}(T) \simeq \underline{X}(T) \simeq Y(T).$$

Conversely, if Y satisfies (2), then by the result we are going to prove, $\underline{Y(*)}$ also satisfies (2) and thus $\underline{Y(*)} \simeq Y$, so Y is discrete.

We now turn to proving the equivalence $\tilde{X} \to \underline{X}$, and start by showing that $\tilde{X}$ is a hypercomplete sheaf.

Since the property of being a hypercomplete sheaf is preserved under limits, writing $X = \varprojlim_n \tau_{\leq n} X$ we reduce to the case where X is a truncated anima. By [Lur09, Lemma 6.5.2.9], it now suffices to show that $\tilde{X}$ is a sheaf.

Lemma (3) gives, for any two sheaves $\mathcal{F}, \mathcal{G}$ on the topological space T, an equivalence

$$\operatorname{Map}_{\operatorname{Sh}(T)}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim_{[n] \in \Delta} \operatorname{Map}_{\operatorname{Sh}\left(S^{\times_T [n]}\right)}(f_n^* \mathcal{F}, f_n^* \mathcal{G})$$

where $f_n : S^{\times_T [n]} = S \times_T \cdots \times_T S \to T$ is the obvious map (this is a general fact about how mapping anima are computed in a limit of ∞-categories). For an anima Y and topological space Z, denote by $\underline{Y}_Z$ the constant sheaf at Y on Z. Applying the above equivalence to $\mathcal{F} = \underline{*}_T$ and $\mathcal{G} = \underline{X}_T$ we get an equivalence

$$\underline{X}_T(T) \simeq \varprojlim_{[n] \in \Delta} \underline{X}_{S^{\times_T [n]}}(S^{\times_T [n]})$$

so in order to show that $\tilde{X}$ is a sheaf, it suffices to show that for every profinite set T, $\underline{X}_T(T) \simeq \tilde{X}(T)$.

Now we fix the topological space T, which is assumed to be profinite, and throughout the rest of this proof, the word (pre)sheaf will mean (pre)sheaf of anima on the topological space T.

A subbasis for the topology on $T = \varprojlim_i T_i$ is given by the clopen sets $B_{t_i} = p_i^{-1}\{t_i\}$ for each $t_i \in T_i$, where $p_i : T \to T_i$ denotes the projection.

Consider the plus construction ([Lur09, Construction 6.2.2.9]) on the constant presheaf X_T. From [Lur09, 6.2.2.9–12], it follows that it is given at a clopen subset $V \subset T$ (which is again profinite, and we can write $V = \varprojlim_i V_i$) by

$$X_T^\dagger(V) = \varinjlim_{R \in J(V)} \operatorname{Hom}_{\operatorname{PSh}(T)}(R, X)$$

where $J(V)$ denotes the poset of covering sieves on V. For a subset $J \subset I$, denote by $R_{J,V}$ the sieve in $J(V)$ generated by the family

$$\left(\bigcap_{j \in J} (B_{t_j} \cap V) \right)_{j \in J, t_j \in T_j}$$

Covering sieves here correspond to collections of open subsets of V that cover V, and a sieve generated by a family F corresponds to the collection of open subsets U of V such that there exists an open $W \in F$ such that $W \subset U$. We note that every covering sieve is of the form $R_{J,V}$ for $J \subset I$ finite, as these are generated by the basis elements. Therefore,

$$X_T^\dagger(V) = \varinjlim_{\substack{J\subset I\\ \text{finite}}} \mathrm{Hom}_{\mathrm{PSh}(T)}(R_J, X_T) \simeq \varinjlim_{\substack{J\subset I\\ \text{finite}}} \prod_{\substack{t_j\in V_j,\\ j\in J}} X \simeq \varinjlim_i \prod_{t_i \in V_i} X \simeq \tilde{X}(V)$$

In the penultimate step, we used that I is filtered. To prove that $X_T^\dagger = \underline{X}_T$, it suffices to show that $X_T^\dagger$ is a sheaf, since this will prove that it satisfies the universal property of the sheafification of the constant presheaf X_T (see [Lur09, Lemma 6.2.2.14]).

We turn to proving that $X_T^\dagger$ is a sheaf. Since the topological space T is coherent (see [Lur09, above Proposition 6.5.4.4]; the compact open subsets are the same as the clopen subsets in this case, since T is a compact Hausdorff space; the clopens are stable under finite intersections and form a basis for the topology on T), it suffices by [Lur09, Theorem 7.3.5.2] to show that $X_T^\dagger(\varnothing) \simeq *$ and that for every pair of clopen subsets $U, V \subset T$, the diagram

$$\begin{array}{ccc} X_T^\dagger(U\cap V) & \longrightarrow & X_T^\dagger(U) \\ \downarrow & & \downarrow \\ X_T^\dagger(V) & \longrightarrow & X_T^\dagger(U\cup V) \end{array}$$

is a pullback square. The condition on the empty set is clear since it is mapped to an empty product, which is equivalent to a point, and the condition on the square above follows from the fact that filtered colimits commute with finite limits.

We have now established that $\tilde{X}$ is a hypercomplete sheaf. It is now simple to prove the desired equivalence: We show that for any extremally disconnected set T,

$$\underline{X}(T) \simeq \underline{X}_T(T).$$

We denote by $\underline{X}^T$ the presheaf on the extremally disconnected set T defined by its values on a basis of clopen subsets by $\underline{X}^T(U) = \underline{X}(U)$. This is a sheaf since it takes finite disjoint unions to finite products (because the clopen subsets are extremally disconnected, and $\underline{X}$ is a sheaf on the site of extremally disconnected sets). Further, it receives a map from the presheaf X_T (since X_T is given at the clopen subsets of T by restriction of the constant presheaf X on the site of extremally disconnected sets, and the sheafification of a presheaf receives a map from that presheaf). It follows that there is a map

$$\underline{X}_T \to \underline{X}^T,$$

which turns out to be an equivalence: indeed, this can be checked on stalks, where it becomes the identity map $X \to X$, and so we are done. □

(8) Corollary. Let Y be a condensed spectrum. The following are equivalent

(1) Y is discrete

(2) For every profinite set $T = \varprojlim_i T_i$, the canonical map

$$\varinjlim_i \prod_{t\in T_i} Y(\{t\}) \simeq \varinjlim_i Y(T_i) \to Y(T)$$

is an equivalence.

Proof. Let X be a spectrum and define $\tilde{X}$ like in the proof of (7). The only difficult thing to prove is that $\tilde{X}$ is a hypercomplete sheaf, but for that it is enough to prove that for every spectrum Y, the presheaf of anima

$$T \mapsto \mathrm{Map}(Y, \tilde{X}(T))$$

is a hypercomplete sheaf. It suffices to check this on the (compact) generators of Sp; namely shifts of the sphere spectrum. But by compactness of Y we have that the presheaf above is actually equivalent to

$$\widetilde{\mathrm{Map}(Y, X)}$$

which is a hypercomplete sheaf by the proof of (7). Now the proof is the same as for anima. □

(9) Corollary. A condensed anima or spectrum X is discrete if and only if for every extremally disconnected set T and every $t \in T$, the canonical map

$$\varinjlim_{U \ni t} X(U) \to X(\{t\}),$$

where the colimit is taken over all clopen neighbourhoods U of t, is an equivalence.

Proof. Let X be an anima or spectrum. As we have seen, $\underline{X}(U) = \underline{X}_T(U)$ for any clopen subset U of T. Thus the left hand side in the desired equivalence is the stalk of $\underline{X}_T$ at t, i.e. X, as is the right hand side.

We conclude that $\underline{X}$ satisfies the desired equivalence, and the result follows by similar arguments to those we have already seen. □

References

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