The Foundations of Condensed Mathematics

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Introduction

Background

Condensed mathematics is a recent effort by Dustin Clausen and Peter Scholze, to resolve some foundational problems with doing algebra when the algebraic structures in question carry a topology. One of the biggest problems is that the category of topological abelian groups is not an abelian category; indeed, there are numerous examples of continuous group homomorphisms that are both monomorphisms and epimorphisms, but not isomorphisms, one simple example being the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$. To resolve this, a notion of *condensed set* is defined, which also extends to *condensed abelian groups, modules, rings* etc. These are sheaves on a certain site of compact Hausdorff spaces. Condensed sets include most nice topological spaces, more precisely the *compactly generated* ones. Condensed abelian groups do form an abelian category, in fact a very nice one. It satisfies all the same of Grothendieck's AB axioms as the category of abelian groups.

Classical notions of cohomology on compact Hausdorff spaces have been extended to cohomology internal to the topos of condensed sets. One application of this is that one can retrieve the notion of derived hom on the bounded derived category of locally compact abelian groups due to Hoffmann and Spitzweck [HS07], by calculating derived internal homs in the much nicer derived category of condensed abelian groups. These are the applications discussed in the present work, but condensed sets have a variety of other applications. So far, they have mostly been explored in the context of analytic geometry: using condensed sets, a new notion of analytic space can be defined, which simultaneously generalises scheme theory, complex analytic geometry, rigid analytic geometry, and real manifolds, and simplifies and unifies some imoprtant thoerems in these subjects (see [Sch19b], [Sch19a]).

Structure

This thesis deals with the foundations of condensed mathematics as presented in the first four lectures of [Sch19b]. In each chapter, we discuss in some detail the prerequisites for the corresponding lecture in [Sch19b], before presenting the main material. Some proofs are given in more detail or slightly altered, but all the main ideas are the same. In particular, I claim no originality of the results. It is my hope, however, that I have succeded in creating a more accessible presentation of the foundations of condensed mathematics, along with the prerequisites, than has previously been available.

Chapter 1 has two sections; in section 1.1 we define the sieve-theoretic notion of sites and sheaves on them, and in section 1.2, three equivalent definitions of condensed sets are given. It is essential in this context to use the sieve-theoretic definition of a site; the definition using fibre products that algebraic geometers usually get away with using is not always enough in our situation.

In chapter 2, we study abelian categories, and their derived categories, in general (sections 2.1, 2.3, 2.4, 2.5, and 2.7) and the category of condensed abelian groups and its derived category (sections 2.2 and 2.6). Although it soon becomes essential to phrase the derived theory of condensed abelian groups in the language of ∞-categories as one progresses further, it is not necessary for the foundations presented in this thesis. Therefore, I have opted to stay within the familiar realm of 1-categories throughout the thesis, and define derived categories as triangulated 1-categories instead of stable ∞-categories.

Chapter 3 is all about cohomology. We begin by discussing simplicial methods at length in section 3.1, to arrive at the notion of cohomology internal to the topos of condensed sets. In section 3.2, this cohomology is compared to the classical notions of sheaf and Čech cohomology on compact Hausdorff spaces.

The derived theory of chapter 2, and the cohomology theory of chapter 3, are then applied to locally compact abelian groups (regarded as condensed abelian groups) in chapter 4. In section 4.1 we briefly introduce topological abelian groups and in particular locally compact abelian groups, and discuss further the relationship between topological spaces and condensed sets. Section 4.2 introduces (very briefly) the necessary results about spectral sequences used in section 4.3 to compute derived internal homs between locally compact abelian groups in the condensed setting.

Acknowledgement

Firstly, I would like to thank Dustin Clausen for accepting to supervise this project when I was applying to become his PhD student. This has assured me that we work well together and that I will arrive in Copenhagen this autumn well equipped to start my PhD on condensed mathematics under his supervision. Dustin has been generous with his time and listened to my questions and doubts at our meetings every week, providing insightful answers and encouraging words.

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Chapter 1

Sites, sheaves, and condensed sets

In this chapter, we explain and lay the groundwork for the different equivalent definitions of condensed sets. We present several equivalent notions of sheaves on a site in section 1.1, preparing to deduce the concrete descriptions of condensed sets given in section 1.2. The definition of condensed sets and the results presented here are due to Clausen and Scholze [Sch19b].

1.1 Sites and sheaves

1.1.1 Sieves and the sheaf condition

(1.1.1) Definition.

- Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$. A *sieve* on X is a subfunctor S of $\mathrm{Hom}(-, X)$, i.e. for all objects Y of $\mathcal{C}$,
$$S(Y) \subset \mathrm{Hom}(Y, X)$$
and if $f \in S(Y)$ then for any $g : Z \to Y$, $f \circ g \in S(Z)$.

- If $f : Y \to X$ is any morphism we define the *pullback of S along f* to be the following sieve f^*S on Y defined by
$$f^*S(Z) = \{g : Z \to Y \mid f \circ g \in S(Z)\}.$$

- Let $F = \{f_i : X_i \to X\}_{i \in I}$ be a family of morphisms with fixed target X. The sieve S *generated by* F is defined by
$$S(Y) = \{f : Y \to X \mid f \text{ factors through some } f_i \in F\}.$$

(1.1.2) Definition. A *presheaf of sets* (or simply *presheaf*) on a category $\mathcal{C}$ is a functor

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}.$$

For a morphism $f : U \to V$ in $\mathcal{C}$, the morphism $\mathcal{F}(f) : \mathcal{F}(V) \to \mathcal{F}(U)$ is often denoted by f^*. For $s \in \mathcal{F}(V)$, $f^*(s)$ is then called the *pullback of s via f*. A *morphism of presheaves* is a natural transformation of functors.

(1.1.3) Definition. Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$, and $\mathcal{F}$ a presheaf on $\mathcal{C}$. For a morphism f, we denote by $\operatorname{dom} f$ its domain.

- Let S be a sieve on X. A family $(x_f)_{f \in S}$ where $x_f \in \mathcal{F}(\operatorname{dom} f)$ is called a *matching family* for the sieve S if for all $f \in S$ and all morphisms g into $\operatorname{dom} f$, we have

  $$g^*(x_f) = x_{f \circ g}.$$

  An *amalgamation* for the matching family is an element $x \in \mathcal{F}(X)$ such that

  $$f^*(x) = x_f$$

  for all morphisms f in S.

- Let

  $$F = \{f_i : X_i \to X\}_{i \in I}$$

  be a family of morphisms with target X. A *matching family* for F is a family $(x_i)_{i \in I}$, with $x_i \in \mathcal{F}(X_i)$ for all i, such that for any commutative diagram

  $$\begin{array}{ccc} Y & \xrightarrow{h} & X_j \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_j} \\ X_i & \xrightarrow[f_i]{} & X \end{array}$$

  we have $g^*(x_i) = h^*(x_j)$. An *amalgamation* for the matching family is an $x \in \mathcal{F}(X)$ such that

  $$f_i^*(x) = x_i$$

  for all $i \in I$.

(1.1.4) Proposition. Let

$$F = \{f_i : X_i \to X\}_{i \in I}$$

be a family of morphisms with target X. Suppose the fibre products $X_i \times_X X_j$ exist for all $i, j \in I$. We want to show that a family $(x_i)_{i \in I}$ with $x_i \in \mathcal{F}(X_i)$ for all $i \in I$ is a matching family for F if and only if

$$\pi_{ij,1}^*(x_i) = \pi_{ij,2}^*(x_j)$$

for all $i, j \in I$, where

$$\pi_{ij,1} : X_i \times_X X_j \to X_i \text{ and } \pi_{ij,2} : X_i \times_X X_j \to X_j$$

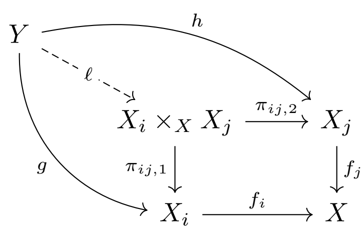
are the projections.

Proof. Let $(x_i)_{i \in I}$ be a matching family. Then $\pi_{ij,1} : X_i \times_X X_j \to X_i$ and $\pi_{ij,2} : X_i \times_X X_j \to X_j$ satisfy the condition imposed on g and h in the definition and it is clear that $\pi^*_{ij,1}(x_i) = \pi^*_{ij,2}(x_j)$ for all $i, j \in I$.

Conversely, let $(x_i)_{i \in I}$ be a family with $x_i \in \mathcal{F}(X_i)$ such that

$$\pi^*_{ij,1}(x_i) = \pi^*_{ij,2}(x_j) \text{ for all } i, j \in I.$$

Suppose $g : Y \to X_i$ and $h : Y \to X_j$ satisfy $f_i \circ g = f_j \circ h$. By the pullback diagram



we have

$$g^*(x_i) = \ell^*(\pi^*_{ij,1}(x_i)) = \ell^*(\pi^*_{ij,2}(x_j)) = h^*(x_j)$$

as desired. □

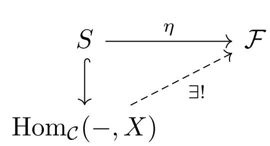
(1.1.5) Theorem. Let $\mathcal{C}$ be a category, X an object of $\mathcal{C}$ and $\mathcal{F}$ a presheaf on $\mathcal{C}$. Let S be a sieve on X. The following conditions are equivalent.

(i) The map

$$\mathrm{Nat}(\mathrm{Hom}(-, X), \mathcal{F}) \to \mathrm{Nat}(S, \mathcal{F})$$

induced by the inclusion $S \hookrightarrow \mathrm{Hom}(-, X)$ is a bijection.

(ii) Every natural transformation $\eta : S \to \mathcal{F}$ has a unique extension to a natural transformation $\mathrm{Hom}(-, X) \to \mathcal{F}$:



(iii) Every matching family for S has a unique amalgamation.

(iv) The diagram

$$\mathcal{F}(X) \xrightarrow{e} \prod_{f \in S} \mathcal{F}(\operatorname{dom} f) \overset{a}{\underset{p}{\rightrightarrows}} \prod_{\substack{f,g \quad f\in S \\ \operatorname{dom} f = \operatorname{cod} g}} \mathcal{F}(\operatorname{dom} g)$$

is an equaliser diagram (i.e. the map e sends $\mathcal{F}(X)$ to the subset of the priduct in the middle on which the maps a and p are equal). The maps are defined as follows: for $x \in \mathcal{F}(X)$, we let $e(x) = (f^*(x))_{f\in S}$. For $\mathbf{x} = (x_f)_{f\in S} \in \prod_{f\in S} \mathcal{F}(\operatorname{dom} f)$,

$$p(\mathbf{x})_{f,g} = x_{f\circ g}, \qquad a(\mathbf{x})_{f,g} = g^*(x_f)$$

If S is generated by a family $F = \{f_i : X_i \to X\}_{i\in I}$, then the conditions above are equivalent to the following as well

(v) Every matching family for F has a unique amalgamation.

If, further, the relevant fibre products exist, the conditions above are equivalent to:

(vi) The diagram

$$\mathcal{F}(X) \xrightarrow{e} \prod_{i\in I} \mathcal{F}(X_i) \overset{p_1}{\underset{p_2}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(X_i \times_X X_j)$$

is an equaliser diagram. The maps are defined as follows: for $x \in \mathcal{F}(X)$,

$$e(x) = (f_i^*(x))_{i\in I};$$

for $\mathbf{x} = (x_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(X_i)$,

$$p_1(\mathbf{x})_{i,j} = \pi^*_{ij,1}(x_i), \qquad p_2(\mathbf{x})_{i,j} = \pi^*_{ij,2}(x_j)$$

where $\pi_{ij,1} : X_i \times_X X_j \to X_i$ and $\pi_{ij,2} : X_i \times_X X_j \to X_j$ are the projections.

Proof. The arguments used in this proof are taken from [Joh02] and [MM94].

Since (ii) is just a reformulation of (i), we immediately have (i) $\iff$ (ii).

Suppose (ii) holds. We want to prove (iii). Let $(x_f)_{f\in S}$ be a matching family. For any object Y and $f \in S(Y)$, define $\eta(f) = x_f$. This gives a natural transformation $\eta : S \to \mathcal{F}$, since

$$\eta(f \circ g) = x_{f\circ g} = g^*(x_f) = g^*(\eta(f)).$$

By (ii), η has a unique extension $\tilde{\eta}$ to $\mathrm{Hom}(-, X)$. Let $x = \tilde{\eta}(\mathrm{id}_X) \in \mathcal{F}(X)$. Then for all Y and all $f \in S(Y)$,

$$f^*(x) = f^*(\tilde{\eta}(\mathrm{id}_X)) = \tilde{\eta}(\mathrm{id}_X \circ f) = \tilde{\eta}(f) = \eta(f) = x_f$$

so x is an amalgamation for the matching family. As for the uniqueness, suppose there is another $x' \in \mathcal{F}(X)$ such that $f^*(x') = x_f$ for all f in S. Let

$$\eta' : \mathrm{Hom}_{\mathcal{C}}(-, X) \to \mathcal{F}$$

be the corresponding (via the Yoneda bijection) natural transformation, i.e. such that $x' = \eta'(\mathrm{id}_X)$. Then for all f in S,

$$\eta'(f) = \eta'(\mathrm{id}_X \circ f) = f^*(\eta'(\mathrm{id}_X)) = f^*(x') = x_f = \eta(f).$$

Thus η' is another natural transformation extending η, so by uniqueness of $\tilde{\eta}$, we have $\eta' = \tilde{\eta}$ and thus $x' = x$.

For the converse, suppose that (iii) holds. We want to prove (ii). Let $\eta : S \to \mathcal{F}$ be a natural transformation. For any object Y and $f \in S(Y)$, let $x_f = \eta(f)$. By naturality of η, this gives a matching family $(x_f)_{f \in S}$ for S. It has a unique amalgamation x, which by a similar argument to the one above (via Yoneda), gives a unique extension $\tilde{\eta}$ of η to $\mathrm{Hom}(-, X)$.

Now (iv) is just a reformulation of (iii), so we are done with the equivalence of the first four conditions.

Suppose then that S is generated by the family $F = \{f_i : X_i \to X\}_{i \in I}$. By proposition (1.1.4), (vi) is just a reformulation of (v) in the case that the relevant fibre products exist, it suffices to show that (iii) and (v) are equivalent.

Suppose that (iii) holds. Let $(x_i)_{i \in I}$ be a matching family for F. Define a matching family $(y_g)_{g \in S}$ for S as follows:

$$y_g = h^*(x_i)$$

where h and i are such that $g = f_i \circ h$. This is well defined (does not depend on the choice of i and h) by the definition of a matching family.

The family $(y_g)_{g \in S}$ is indeed a matching family because if k is any morphism such that the compositions make sense, we have

$$k^*(y_g) = k^*(h^*(x_i)) = (h \circ k)^*(x_i) = y_{g \circ k}$$

because $g \circ k = f_i \circ h \circ k$.

By (iii), this matching family has a unique amalgamation $y \in \mathcal{F}(X)$, i.e. an element satisfying $g^*(y) = y_g$ for all $g \in S$. In particular, for all $i \in I$,

$$f_i^*(y) = y_{f_i} = x_i$$

so y is an amalgamation for the family $(x_i)_{i\in I}$. As for uniqueness, suppose that there is another $y' \in \mathcal{F}(X)$ such that $f_i^*(y') = x_i$ for all i. For $g \in S$, write $g = f_i \circ h$. Then

$$g^*(y') = h^*(f_i^*(y')) = h^*(x_i) = y_g$$

so $y' = y$ by uniqueness of the amalgamation for the family (y_g).

Finally, suppose that (v) holds. We want to show (iii). Let $(y_g)_{g\in S}$ be a matching family for S. Then the subfamily $(y_{f_i})_{i\in I}$ is a matching family for F: indeed, let $g : Y \to X_i$ and $h : Y \to X_j$ be such that $f_i \circ g = f_j \circ h$. Then

$$g^*(y_{f_i}) = y_{f_i\circ g} = y_{f_j\circ h} = h^*(y_{f_i}).$$

Thus there exists a unique $y \in \mathcal{F}(X)$ such that $f_i^*(y) = y_{f_i}$ for all i. For $g \in S$, write $g = f_i \circ h$. Then

$$g^*(y) = (f_i \circ h)^*(y) = h^*(f_i^*(y)) = h^*(y_{f_i}) = y_{f_i\circ h} = y_g$$

where the second-to-last equality is just because (y_g) is a matching family. We conclude that the matching family $(y_g)_{g\in S}$ has the unique amalgamation y as well. □

(1.1.6) Definition. If a presheaf $\mathcal{F}$ satisfies one of the equivalent conditions in theorem (1.1.5), we say that $\mathcal{F}$ *satisfies the sheaf condition with respect to the sieve* S.

1.1.2 Sites and sheaves

(1.1.7) Definition. Let $\mathcal{C}$ be a category. A *coverage* τ on $\mathcal{C}$ is given by specifying a set $\mathrm{Cov}_\tau(X)$ of *covering sieves* for each object X, satisfying

- If $S \in \mathrm{Cov}_\tau(X)$ and $f : Y \to X$, then there is a sieve $R \subset f^*S$ such that $R \in \mathrm{Cov}_\tau(Y)$.

The pair $(\mathcal{C}, \tau)$ is called a *site*.

(1.1.8) Definition. A presheaf $\mathcal{F}$ on a site $(\mathcal{C}, \tau)$ is called a *sheaf* if it satisfies the sheaf condition (see theorem (1.1.5)) with respect to every covering sieve.

(1.1.9) Definition. A *Grothendieck pretopology* $\mathcal{P}$ on a category $\mathcal{C}$ is given, for each object X of $\mathcal{C}$, by a set $\mathrm{Cov}_\mathcal{P}(X)$ of families $\{X_i \to X\}_{i\in I}$ of morphisms, satisfying

(1) If $Y \to X$ is an isomorphism then $\{Y \to X\} \in \mathrm{Cov}_\mathcal{P}(X)$.

(2) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}_\mathcal{P}(X)$ and $\{Y_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}_\mathcal{P}(X_i)$ for all $i \in I$, then the family of compositions $\{Y_{ij} \to X\}_{i\in I, j\in J_i} \in \mathrm{Cov}_\mathcal{P}(X)$

(3) If $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}_{\mathcal{P}}(X)$ and $Y \to X$ is a morphism of $\mathcal{C}$ then $X_i \times_X Y$ exists for all i and $\{X_i \times_X Y \to Y\}_{i\in I} \in \mathrm{Cov}_{\mathcal{P}}(Y)$.

(1.1.10) Example. Let X be a topological space and $\mathcal{C} = \mathrm{Op}_X$ be the category whose objects are the open subsets of X and whose arrows are the inclusions. Then we obtain a natural Grothendieck pretopology $\mathcal{P}$ on $\mathcal{C}$ by defining covering families as follows

$$\{U_i \to U\} \in \mathrm{Cov}_{\mathcal{P}}(U) \iff \bigcup_{i\in I} U_i = U.$$

Then one automatically has (1) and (2). Since $U_i \times_U V = U_i \cap V$ in this context, (3) follows.

(1.1.11) Definition. Let $\mathcal{C}$ be a category. A *Grothendieck topology* $\mathcal{T}$ on $\mathcal{C}$ is given by specifying a set $\mathrm{Cov}_{\mathcal{T}}(X)$ of *covering sieves* for each object X, satisfying

(1) (Identity) For all objects X, $\mathrm{Hom}(-, X) \in \mathrm{Cov}_{\mathcal{T}}(X)$.

(2) (Base change) If $S \in \mathrm{Cov}_{\mathcal{T}}(X)$ and $f : Y \to X$, then $f^*S \in \mathrm{Cov}_{\mathcal{T}}(Y)$.

(3) (Local character) If $S \in \mathrm{Cov}_{\mathcal{T}}(X)$ and R is a sieve on X such that

$$f^*R \in \mathrm{Cov}_{\mathcal{T}}(Y)$$

for all objects Y and all $f \in S(Y)$, then $R \in \mathrm{Cov}_{\mathcal{T}}(X)$.

(1.1.12) Definition. Let $\mathcal{P}$ be a Grothendieck pretopology on a category $\mathcal{C}$. The *Grothendieck topology generated by* $\mathcal{P}$, here denoted $\mathcal{T}$, is defined as follows. For any object X of $\mathcal{C}$, let $S \in \mathrm{Cov}_{\mathcal{T}}(X)$ if and only if the sieve S contains some family from $\mathrm{Cov}_{\mathcal{P}}(X)$.

(1.1.13) Remark. We need to show that $\mathcal{T}$ as in (1.1.12) is a Grothendieck topology. Let X be an object of $\mathcal{C}$. Then

$$\mathrm{Hom}(-, X) \in \mathrm{Cov}_{\mathcal{T}}(X)$$

since it contains the family $\{\mathrm{id}_X\} \in \mathrm{Cov}_{\mathcal{P}}(X)$.

For the second condition, let $f : Y \to X$ and $S \in \mathrm{Cov}_{\mathcal{T}}(X)$. Let

$$\{X_i \to X\}_{i\in I} \in \mathrm{Cov}_{\mathcal{P}}(X)$$

contained in S. Then the relevant fibre products exist and

$$\{X_i \times_X Y \to Y\} \in \mathrm{Cov}_{\mathcal{P}}(Y).$$

Also

$$\{X_i \times_X Y \to Y\} \subset f^*S$$

so $f^*S \in \mathrm{Cov}_{\mathcal{T}}(Y)$.

Finally, suppose S contains a family $\{f_i : X_i \to X\}_{i \in I}$ and R is a sieve on X satisfying the property that for all Y, for all $f \in S(Y)$, $f^*R \in \mathrm{Cov}_{\mathcal{T}}(Y)$. Then in particular, for all $i \in I$ there exists a family

$$\{Y_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}_{\mathcal{P}}(X_i)$$

contained in f_i^*R. Then the family of compositions

$$\{Y_{ij} \to X\}_{i \in I, j \in J_i}$$

is in $\mathrm{Cov}_{\mathcal{P}}(X)$ and contained in R, so $R \in \mathrm{Cov}_{\mathcal{T}}(X)$.

(1.1.14) Definition. Let τ be a coverage on a category $\mathcal{C}$. The *Grothendieck topology* $\mathcal{T}$ *generated by* τ is defined as the intersection of all Grothendieck topologies containing τ. More precisely, the set $\mathrm{Cov}_{\mathcal{T}}(X)$ of covering sieves on X for $\mathcal{T}$ is the intersection $\bigcap \mathrm{Cov}_{\mathcal{T}'}(X)$ where $\mathcal{T}'$ ranges over all Grothendieck topologies such that every covering sieve for τ is a covering sieve for $\mathcal{T}'$.

(1.1.15) Remark. It is clear from the axioms for a Grothendieck topology that the type of intersection of Grothendieck topologies used in definition (1.1.14) is itself a Grothendieck topology.

(1.1.16) Remark. A Grothendieck topology $\mathcal{T}$ is a coverage by axiom (2) in the definition. The Grothendieck topology generated by the coverage $\mathcal{T}$ is clearly $\mathcal{T}$ itself.

(1.1.17) Remark. A coverage and the Grothendieck topology it generates have the same sheaves, as lemmas (1.1.18), (1.1.19) and (1.1.20) show. This shows that axioms (1) and (3) don't have any effect on which presheaves are sheaves; axiom (2) is the most important one.

(1.1.18) Lemma. Let $\mathcal{F}$ be a presheaf on a category $\mathcal{C}$ and let X be an object of $\mathcal{C}$. Then $\mathcal{F}$ satisfies the sheaf condition with respect to the sieve $\mathrm{Hom}(-, X)$

Proof. This is immediate, for example by using the sheaf condition (i) in theorem (1.1.5). □

(1.1.19) Lemma. Let $\mathcal{F}$ be a sheaf on a site $(\mathcal{C}, \tau)$. Then $\mathcal{F}$ satisfies the sheaf condition with respect to any sieve S which contains a τ-covering sieve R.

Proof. Let $(x_f)_{f \in S}$ be a matching family for S. Then $(x_f)_{f \in R}$ is a matching family for R which by assumption has a unique amalgamation $x \in \mathcal{F}(X)$ (S and R are sieves on X). We want to show that x is an amalgamation for S. Let $f \in S(Y)$. Then f^*R contains a τ-covering sieve T of Y. Define a matching family $(y_k)_{k\in T}$ for T by $y_k = x_{f\circ k}$. Then $k^*(x_f) = x_{f\circ k} = y_k$ so x_f is the unique amalgamation for (y_k). Since $f \circ k \in R$, we further have that

$$y_k = x_{f\circ k} = (f \circ k)^*(x) = k^*(f^*(x))$$

so $f^*(x)$ is another amalgamation for (y_k). By uniqueness, $f^*(x) = x_f$, showing that x is an amalgamation for $(x_f)_{f\in S}$. Uniqueness is clear because of uniqueness for $(x_f)_{f\in R}$. □

(1.1.20) Lemma. Let $\mathcal{F}$ be a sheaf on a site $(\mathcal{C}, \tau)$. Let S be a τ-covering sieve on an object X of $\mathcal{C}$ and R another sieve on X such that for all $f \in S$, f^*R is a τ-covering sieve of $\operatorname{dom} f$. Then $\mathcal{F}$ satisfies the sheaf condition with respect to R.

Proof. Let's first show that $\mathcal{F}$ satisfies the sheaf axiom for the sieve T on X consisting of the composites $f\circ h$ where $f \in S$ and $h \in f^*R$. Let $(x_{f\circ h})_{f\in S, h\in f^*R}$ be a matching family for T. Fix an $f \in S$ and let $y_h = x_{f\circ h}$. Then $(y_h)_{h\in f^*R}$ is a matching family for f^*R and so has a unique amalgamation which we denote z_f. This defines a matching family $(z_f)_{f\in S}$ which then has a unique amalgamation $x \in \mathcal{F}(X)$. Clearly, x is also an amalgamation for $(x_{f\circ h})$ defined above. For uniqueness, suppose that x' is another amalgamation. Let $f \in S$. Then for all $h \in f^*R$,

$$h^*(f^*(x')) = (f \circ h)^*(x') = x_{f\circ h} = y_h$$

and by uniqueness of z_f, we have $f^*(x') = z_f$. This holds for all $f \in S$, and by uniqueness of x, we conclude that $x' = x$.

Now we show that $\mathcal{F}$ satisfies the sheaf condition with respect to R. We first show that $T = R \cap S$. It is clear that $T \subset R$ and $T \subset S$. If $g \in R \cap S$, then $g^*R = \operatorname{Hom}(-, \operatorname{dom} g)$ and we can write $g = f \circ h$ with $f = g \in S$ and $h = \mathrm{id}_{\operatorname{dom} g} \in g^*R$.

So let $(x_g)_{g\in R}$ be a matching family for R. Then $(x_g)_{g\in T}$ is a matching family for T with unique amalgamation $x \in \mathcal{F}(X)$. Let $g \in R$. Then g^*S contains a τ-covering sieve, and by lemma (1.1.19), $\mathcal{F}$ satisfies the sheaf condition with respect to $g^*S = g^*(R \cap S) = g^*T$. Define a matching family $(y_k)_{k\in g^*T}$ for g^*T by $y_k = x_{g\circ k}$. Then both x_g and $g^*(x)$ are amalgamations for this family (same argument as in the proof of lemma (1.1.19)) and we conclude that x is an amalgamation for $(x_g)_{g\in R}$; uniqueness is clear. □

(1.1.21) Remark. As theorem (1.1.5) shows, it is sometimes easier to work with families generating sieves than the sieves themselves, which is the motivation behind Definition (1.1.22).

(1.1.22) Definition. Let $\mathcal{C}$ be a category. A *precoverage* π on $\mathcal{C}$ is given by specifying a set $\operatorname{Cov}_\pi(X)$ of *covering families* $\{X_i \to X\}_{i\in I}$ of morphisms with target X, satisfying the following condition

- If $\{f_i : X_i \to X\}_{i\in I} \in \mathrm{Cov}_\pi(X)$ and $g : Y \to X$ is any morphism, then there exists a $\{h_j : Y_j \to Y\}_{j\in J} \in \mathrm{Cov}_\pi(Y)$ such that each $g \circ h_j$ factors through an f_i.

(1.1.23) Remark. A coverage is just a precoverage for which every covering family is a sieve:

Proof. Let τ be a coverage. Let $S \in \mathrm{Cov}_\tau(X)$ be a covering sieve and let $g : Y \to X$ be a morphism. Then g^*S contains an $R \in \mathrm{Cov}_\tau(Y)$, and for every $h \in R$, $g \circ h \in S$, in particular factors through a morphism in S. This proves that τ is a precoverage. Conversely, suppose we have a precoverage π for which every covering family is a sieve. Then let $S \in \mathrm{Cov}_\tau(X)$ and let $g : Y \to X$ be any morphism. Let $R = \{h_j : Y_j \to Y\}_{j\in J} \in \mathrm{Cov}_\pi(Y)$ be such that each $g \circ h_j$ factors through an $f \in S$. Then we can write $g \circ h_j = f \circ k \in S$, and thus $h_j \in g^*S$ for all j, so π is a coverage. □

(1.1.24) Remark. There doesn't seem to be a standard terminology for the different types of coverages defining a site. Sometimes what we call a Grothendieck pretopology is called a Grothendieck topology, what we call a coverage is called a sifted coverage while our precoverages are called coverages. Johnstone [Joh02] avoids the term *topology* altogether and talks about Grothendieck coverages instead of Grothendieck topologies.

(1.1.25) Definition. The *coverage generated by* a precoverage π is the coverage whose covering sieves are those generated by the covering families in π.

(1.1.26) Definition. The *Grothendieck topology generated by* a precoverage is the Grothendieck topology generated by the coverage generated by said precoverage.

(1.1.27) Remark. A Grothendieck pretopology is a precoverage: Indeed, axiom (3) in the definition of a pretopology gives the desired family. We prove here below that the notions of a Grothendieck topology generated by a pretopology in the sense of Definition (1.1.12) and in the sense of Definition (1.1.26) are the same.

Proof. Let $\mathcal{P}$ be a Grothendieck pretopology. Let τ be the coverage generated by the precoverage $\mathcal{P}$ and $\mathcal{T}$ the Grothendieck topology generated by τ. Let $\mathcal{T}'$ be the Grothendieck topology generated by the pretopology $\mathcal{P}$, i.e. the collection of sets $\mathrm{Cov}_{\mathcal{T}'}(X)$ of sieves on X such that each sieve in $\mathrm{Cov}_{\mathcal{T}'}(X)$ contains a family from $\mathrm{Cov}_{\mathcal{P}}(X)$.

Let $S \in \mathrm{Cov}_{\mathcal{T}'}(X)$ and let $F \in \mathrm{Cov}_{\mathcal{P}}(X)$ such that $F \subset S$. Let S' be the sieve generated by F. Then $S' \in \mathrm{Cov}_\tau(X)$ and thus $S' \in \mathrm{Cov}_{\mathcal{T}}(X)$. For any $f \in S' \subset S$, $f^*S = \mathrm{Hom}(-, \mathrm{dom}\, f) \in \mathrm{Cov}_{\mathcal{T}}(\mathrm{dom}\, f)$. Thus $S \in \mathrm{Cov}_{\mathcal{T}}(X)$, so we have $\mathcal{T}' \subset \mathcal{T}$.

We know that $\tau \subset \mathcal{T}'$, and this implies $\mathcal{T} \subset \mathcal{T}'$ and we are done. □

(1.1.28) Corollary. Let $\mathcal{C}$ be a category and $\mathcal{F}$ a presheaf on $\mathcal{C}$. Suppose we have a Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$, generated by a precoverage π. Then to check that $\mathcal{F}$ is a sheaf for $\mathcal{T}$, it suffices to check the sheaf condition on the covering families from π. □

(1.1.29) Sheafification. Let $\mathcal{C}$ be a category equipped with a Grothendieck pretopology $\mathcal{P}$. The inclusion functor $i : \mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ of the category of sheaves on the site $\mathcal{C}$ in the presheaf category, admits a left adjoint $\mathcal{F} \mapsto \mathcal{F}^{\#}$. The sheaf $\mathcal{F}^{\#}$ is called the *sheafification of* $\mathcal{F}$. See [Sta21, Section 00W1], in particular [Sta21, Tags 00WB, 00WG, 00WH].

(1.1.30) Topoi. Condensed sets form a so-called *topos*. The word "topos" is Greek for "place" and Grothendieck thought of a topos as a category that serves as a plave to do mathematics (see [Bae21]). We will not go into the details of topos theory, but we give the definition:

- A *topos* is the category $\mathrm{Sh}(\mathcal{C})$ of sheaves on a site $\mathcal{C}$.
- Let $\mathcal{C}$ and $\mathcal{D}$ be sites. A *morphism of topoi* $f : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ is given by an adjoint pair (f^{-1}, f_*) of functors
$$f^{-1} : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D}), \quad f_* : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C}),$$
i.e.
$$\mathrm{Hom}_{\mathrm{Sh}(\mathcal{D})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(\mathcal{G}, f_*\mathcal{F})$$
bifunctorially in $\mathcal{F}, \mathcal{G}$. Additionally, we impose that f^{-1} commute with finite limits, i.e. is left exact. We define the *composition* $f \circ g$ of two morphisms of topoi $f : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ and $g : \mathrm{Sh}(\mathcal{E}) \to \mathrm{Sh}(\mathcal{D})$ as the morphism of topoi given by $(f \circ g)_* = f_* \circ g_*$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$

1.2 Condensed sets

1.2.1 Sites of compact Hausdorff spaces

(1.2.1) A smallness condition. From now on, we fix an uncountable strong limit cardinal κ. This means that κ has the following property: for any cardinal $\lambda < \kappa$, we have $2^\lambda < \kappa$. Denote by CHaus the category of κ-small (i.e. of cardinality $< \kappa$) compact Hausdorff spaces and continuous maps. We will consider two subcategories of CHaus: Prof and ED (defined below). We will notably be able to take Stone-Čech compactifications without having to worry about our sets becoming so large as to leave CHaus.

(1.2.2) Definition. A *profinite* set is a cofiltered limit of finite sets (viewed as discrete spaces) in the category of topological spaces. See Appendix A for conventions for limits.

(1.2.3) Remark. By [Sta21, Lemma 08ZY], the category of profinite sets is equivalent to the category of totally disconnected compact Hausdorff spaces. We denote the category of κ-small profinite sets by Prof.

(1.2.4) Definition. A topological space S is called *extremally disconnected* if the closure of every open set is open. If S is a compact Hausdorff space, an equivalent definition due to Gleason (see [Gle58]) is the following: every surjection $S' \to S$ from a compact Hausdorff space splits (i.e. has a continuous right inverse).

(1.2.5) Proposition. An extremally disconnected Hausdorff space S is totally disconnected.

Proof. Let $x \in S$ and denote $C(x)$ its connected component. Take any $y \in S$ such that $y \neq x$. We want to show that $y \notin C(x)$. Since S is Hausdorff, we can take an open neighborhood U of x such that $y \notin \overline{U}$. Since S is extremally disconnected, $\overline{U} \cap C(x)$ is open and closed and thus equal to $C(x)$. Therefore, $y \notin C(x)$ and we conclude that $C(x) = \{x\}$, as desired. ☐

(1.2.6) Notation. The category of κ-small extremally disconnected compact Hausdorff spaces will be denoted ED.

(1.2.7) Example. The Stone-Čech compactification βS of a discrete set S is extremally disconnected.

Proof. Indeed, the universal property of the Stone-Čech compactification of S is the following: We have a compact Hausdorff space βS and a dense inclusion $i : S \hookrightarrow \beta S$ such that for any compact Hausdorff space K and continuous $f : S \to K$, there is a unique $\beta f : \beta S \to K$ such that $(\beta f) \circ i = f$.

Now take a compact Hausdorff space S' and surjection $g : S' \to \beta S$. Define $f : S \to S'$ as any set-theoretic right inverse of g (i.e. for any $x \in S$, we pick $f(x)$ to be any y such that $g(y) = x$). This f is automatically continuous since S is discrete, and thus extends uniquely to a $\beta f : \beta S \to S'$. We have $g \circ \beta f \circ i = g \circ f = \mathrm{id}_S$, but since S is dense in βS, we have $g \circ \beta f = \mathrm{id}_{\beta S}$. ☐

(1.2.8) Example. Let S be a compact Hausdorff space and denote by S^δ the set S with the discrete topology. We have that $\mathrm{id}_S : S^\delta \to S$ is continuous and thus we have $\beta\,\mathrm{id}_S : \beta S^\delta \to S$ such that $(\beta\,\mathrm{id}_S) \circ i = \mathrm{id}_S$, thus $\beta\,\mathrm{id}_S$ is a surjection onto S from an extremally disconnected set.

(1.2.9) Remark. We have an inclusion of categories

$$\mathrm{ED} \subset \mathrm{Prof} \subset \mathrm{CHaus}$$

and we want to make each one into a site. First, we need some topological properties of these categories.

(1.2.10) Proposition. Let $\mathcal{C}$ be any one of the three categories in (1.2.9). Then we have the following

(1) For any $S \in \mathcal{C}$ and any closed subspace $R \subset S$, we have $R \in \mathcal{C}$.

(2) Any map $f : S \to R$ in $\mathcal{C}$ is closed.

For $\mathcal{C} = \mathrm{Prof}$ or $\mathcal{C} = \mathrm{CHaus}$, we further have

(3) For maps $f_1 : S_1 \to S$ and $f_2 : S_2 \to S$ in $\mathcal{C}$, the fibre product $S_1 \times_S S_2$ is in $\mathcal{C}$.

Proof. (1) A subspace of a Hausdorff space is Hausdorff. A closed subspace of a compact Hausdorff space is compact. So the statement is true for CHaus. Moreover, a subspace of a totally disconnected space is obviously totally disconnected so the statement is true for Prof. If $S \in \mathrm{ED}$ and $R \subset S$ is closed, we take an open $U \subset R$. There is an open $U' \subset S$ such that $U = R \cap U'$. We want to show that the closure $\overline{U}^R$ of U in R is open in R. Since R is closed in S, we have

$$\overline{U}^R = \overline{U'} \cap R$$

and since $\overline{U'}$ is open in S, this is open in R.

(2) Since the continuous image of a compact space is compact, and a subspace of a compact Hausdorff space is compact if and only if it is closed, this is clear.

(3) It suffices to show that $S_1 \times S_2$ is in $\mathcal{C}$ and that

$$S_1 \times_S S_2 = \{(x, y) \in S_1 \times S_2 \mid f_1(x) = f_2(y)\}$$

is a closed subspace. We know that $S_1 \times S_2$ is compact Hausdorff if S_1 and S_2 are. Suppose S_1 and S_2 are totally disconnected. Since the connected components in a product space are the products of the connected components, this is clear. Since the diagonal is closed in $S \times S$ (because S is Hausdorff), and the fibre product is the preimage of it by the continuous map $S_1 \times S_2 \to S \times S$, $(x, y) \mapsto (f_1(x), f_2(y))$, we are done.

□

(1.2.11) Definition. A family of maps $\{S_i \to S\}_{i\in I}$ is called *jointly surjective* if the induced map

$$\coprod_{i\in I} S_i \to S$$

is surjective.

(1.2.12) Proposition. Let $\mathcal{C}$ be any one of the three categories in remark (1.2.9). Then the finite jointly surjective families in $\mathcal{C}$ form a precoverage. We denote the Grothendieck topology which it generates, by $\mathcal{T}$.

Proof. Let $\{f_i : S_i \to S\}_{i\in I}$ be a finite jointly surjective family. Let $g : R \to S$ be any continuous map.

In the case where $\mathcal{C} = \text{CHaus}$ or $\mathcal{C} = \text{Prof}$, we consider the family

$$\{R \times_S S_i \to R\}_{i\in I}.$$

It is jointly surjective and each member composed with g factors through an f_i by definition of the fibre product.

Now consider the case where $\mathcal{C} = \text{ED}$. Let $R_i := g^{-1}(f_i(S_i))$. Then $R_i \in \mathcal{C}$ for all i because the f_i are closed and g is continuous. Moreover, $R = \bigcup R_i$ and thus the inclusions $R_i \hookrightarrow R$ form a finite jointly surjective family. It suffices to show that each $g_{|R_i}$ factors through f_i. For each i, the map $f_i : S_i \to f_i(S_i)$ is a surjection in ED and thus has a section $h_i : f_i(S_i) \to S_i$. Then $g_{R_i} = f_i \circ h_i \circ g_{|R_i}$ factors through f_i. □

(1.2.13) Proposition. Let $\mathcal{C}$ be as in proposition (1.2.12). Then the the collection of all families of the following two types forms a precoverage on $\mathcal{C}$:

1. $\{f_i : S_i \to S\}_{i\in I}$ such that I is finite and the induced $\coprod_{i\in I} S_i \to S$ is an isomorphism and,
2. singleton families $\{p : S' \to S\}$ where $p : S' \to S$ is surjective.

Moreover, this precoverage generates the same Grothendieck topology $\mathcal{T}$.

Proof. Let $g : R \to S$ be any morphism. The proof of proposition (1.2.12) dealt with the case of families of type 2. For $\{f_i : S_i \to S\}$ a family of type 1, we let $R_i = g^{-1}(f_i(S_i))$. Since f_i is an homeomorphism $S_i \to f_i(S_i)$, a similar argument to the one for ED in proposition (1.2.12) shows that the family of inclusions $\{R_i \to R\}$ has the desired property of the compositions with g factoring through an f_i.

Denote this precoverage by π_1. We want to show that the Grothendieck topology $\mathcal{T}$ is generated by π_1. Denote by π_2 the precoverage from proposition (1.2.12) and by $\mathcal{T}_1$ the Grothendieck topology generated by π_1. We clearly have $\pi_1 \subset \pi_2$, so $\mathcal{T}_1 \subset \mathcal{T}$. Now let $Z \in \mathrm{Cov}_{\mathcal{T}}(S)$ be a sieve on $S \in \mathcal{C}$ generated by a finite jointly surjective family $\{f_i : S_i \to S\}$. Denote by

$$f : \coprod_{i \in I} S_i \to S$$

the map induced by the f_i and for each $j \in I$, let

$$\varphi_j : S_j \hookrightarrow \coprod_{i \in I} S_i$$

be the inclusion. For each i,

$$f \circ \varphi_i = f_i \in Z, \text{ so } \varphi_i \in f^* Z.$$

Therefore,

$$f^* Z \in \mathrm{Cov}_{\mathcal{T}_1}\left(\coprod_{i \in I} S_i\right)$$

and further, for any map h into $\coprod_{i \in I} S_i$,

$$h^*(f^* Z) \in \mathrm{Cov}_{\mathcal{T}_1}\left(\coprod_{i \in I} S_i\right),$$

by the base change axiom for Grothendieck topologies. Denote the sieve generated by $\{f\}$ by Z_f. We have $Z_f \in \mathrm{Cov}_{\mathcal{T}_1}(S)$. For any $g \in Z_f(R)$, write $g = f \circ h$. Then

$$g^* Z = (f \circ h)^* Z = h^*(f^* Z) \in \mathrm{Cov}_{\mathcal{T}_1}\left(\coprod_{i \in I} S_i\right).$$

By the local character of Grothendieck topologies, this implies that $Z \in \mathrm{Cov}_{\mathcal{T}_1}(S)$. We conclude that $\mathcal{T} \subset \mathcal{T}_1$, and we are done. □

1.2.2 Condensed sets

(1.2.14) Definition. A *condensed set* is a sheaf of sets on any of the sites ED, Prof or CHaus, with the covering families given by finite jointly surjective families.

(1.2.15) Remark. theorem (1.2.16) shows that condensed sets (1.2.14) are well defined. theorems (1.2.17) and (1.2.18) give a nice characterisation of condensed sets and help us prove theorem (1.2.16). All three theorems are proved in subsection 1.2.3.

(1.2.16) Theorem. Let $\mathrm{Sh}(\mathcal{C})$ denote the category of sheaves on a site $\mathcal{C}$. The restriction functors

$$\mathrm{Sh}(\mathrm{CHaus}) \to \mathrm{Sh}(\mathrm{Prof}) \to \mathrm{Sh}(\mathrm{ED})$$

are equivalences of categories.

(1.2.17) Theorem. Let $\mathcal{C}$ be one of the sites CHaus or Prof. Then a presheaf T on $\mathcal{C}$ is a sheaf if and only if it satisfies the following two conditions.

(i) For any finite collection $(S_i)_{i\in I}$ of objects of $\mathcal{C}$, the natural map

$$T\left(\coprod_{i\in I} S_i\right) \to \prod_{i\in I} T(S_i)$$

is a bijection.

(ii) For any surjection $S' \to S$ of objects of $\mathcal{C}$, let p_1 and p_2 denote the two projections $S' \times_S S' \to S'$. Then the map

$$T(S) \to \{x \in T(S') \mid p_1^*(x) = p_2^*(x) \in T(S' \times_S S')\}$$

is a bijection.

(1.2.18) Theorem. A presheaf T on ED is a sheaf if and only if it satisfies the following condition

(i) For any finite collection $(S_i)_{i\in I}$ of objects of ED, the natural map

$$T\left(\coprod_{i\in I} S_i\right) \to \prod_{i\in I} T(S_i)$$

is a bijection.

We prove these three theorems in subsection 1.2.3.

(1.2.19) Underlying set and associated condensed set. Given a condensed set T, we call the set $T(*)$ its *underlying set*.

Let T be a topological space. Then the functor $\underline{T}$ sending each profinite set S to the set of continuous maps $S \to T$ is a condensed set. This is clear by the characterisation of condensed sets as sheaves on the site ED (we only need to show that

$$\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$$

which is just the univsersal property of the coproduct). This $\underline{T}$ is called the *condensed set associated to* T.

The functor $T \mapsto \underline{T}$ from topological spaces to condensed set is fully faithful when restricted to a certain subcategory (of so-called *compactly generated* *spaces*). In this case it is also possible to equip the underlying set of any condensed set with a topology in such a way that $T \mapsto T(*)$ becomes left adjoint to the associated condensed set functor. This will not become important until chapter 4 and will be discussed in more detail there.

1.2.3 Proofs

Proof of theorems (1.2.17) and (1.2.18). A *sieve of type 1* will in this proof mean a sieve generated by a finite family $\{S_i \to S\}_{i\in I}$ such that $\coprod_{i\in I} S_i \to S$ is an isomorphism. A *sieve of type 2* will then mean a sieve generated by a singleton family $\{S' \to S\}$ with $S' \to S$ a surjection. We will show that T satisfies (i) if and only if it satisfies the sheaf condition with respect to every sieve of type 1 (in any one of the three sites).

Any sieve of type 2 in ED is of the form $\mathrm{Hom}(-, S)$ (every map into S factors through the surjection $S' \to S$ by composing with its right inverse), and since any presheaf satisfies the sheaf condition with respect to that sieve, this will suffice to prove theorem (1.2.18).

To finish the proof of theorem (1.2.17) we will go on to show that in Prof and CHaus, T satisifies (ii) if and only if it satisfies the sheaf condition with respect to every sieve of type 2.

Now let $\{S_i \to S\}_{i\in I}$ be a family of the first type. We can assume that

$$S = \coprod_{i\in I} S_i$$

and that the maps $S_i \to S$ are the inclusions. Then in all the three sites, the fibre products $S_i \times_S S_j$ exist: if $i = j$ then it is equal to S_i, if not it is empty. Then the two parallel maps in the equaliser diagram in condition (vi) of theorem (1.1.5) are the identity, which means that it is equivalent to

$$T\left(\coprod_{i\in I} S_i\right) = \prod_{i\in I} T(S_i).$$

Now we are done with theorem (1.2.18).

Moving on to any of the sites Prof or CHaus, we see that for sieves of type 2, condition (vi) in theorem (1.1.5) is equivalent to (ii) in (1.2.17), which says that $T(S)$ is the equaliser of the two arrows $T(S') \to T(S' \times_S S')$. □

Proof of theorem (1.2.16). It suffices to show that the value of a sheaf on CHaus is uniquely determined by its restriction to profinite sets, and that the value of a sheaf on Prof is uniquely determined by its restriction to extremally disconnected sets.

For the first statement, let S be a compact Hausdorff space and $S' \to S$ a surjection from a profinite set (for example the Stone-Čech compactification of S viewed as a discrete set). Then by condition (ii) in theorem (1.2.17), $T(S)$ is determined by its values on the profinite sets S' and $S' \times_S S'$.

For the second, let S be a profinite set and $\tilde{S} \to S$ a surjection from an extremally disconnected set (for example the Stone-Čech compactification of S viewed as a discrete set). Further, let $\tilde{\tilde{S}} \to \tilde{S} \times_S \tilde{S}$ be a surjection from an extremally disconnected set. Since $T(\tilde{S} \times_S \tilde{S}) \to T(\tilde{\tilde{S}})$ is injective (it is a bijection onto a subset of the latter by (ii)), $T(S)$ is determined as the equaliser of the two maps $T(\tilde{S}) \to T(\tilde{\tilde{S}})$, i.e. by its values on the two extremally disconnected sets. $\square$

Chapter 2

Condensed abelian groups and their derived category

In this chapter, we prove that the category of condensed abelian groups is a remarkably nice abelian category, in particular generated by compact projectives. In section 2.1, we recall the definition of an abelian category and some of its properties. We move to the condensed setting in section 2.2 where we prove that CondAb satisfies all the same of Grothendieck's axioms for abelian categories as the category of abelian groups, as well as proving that it is generated by compact projectives. Going back to the general setting, we discuss derived categories and derived functors quite extensively in sections 2.3, 2.4, and 2.5. This is all to prepare for going back to the condensed setting and defining the derived category of condensed abelian groups in section 2.6. We end the chapter in section 2.7 with a short discussion about derived limits and colimits, which belongs in this chapter, although it is not used until chapter 4.

2.1 Abelian categories

(2.1.1) Definition. A category $\mathcal{A}$ is called *preadditive* if all the hom sets $\mathrm{Hom}_{\mathcal{A}}(A, B)$ have an abelian group structure, such that all the composition maps

$$\mathrm{Hom}_{\mathcal{A}}(B, C) \times \mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{A}}(A, C)$$

are bilinear. Further, a functor between preadditive categories is called *additive* if the induced maps on hom sets are group homomorphisms.

(2.1.2) Proposition. Let A be an object of a preadditive category $\mathcal{A}$. The following are equivalent.

(i) A is an initial object of $\mathcal{A}$

(ii) A is a final object of $\mathcal{A}$

(iii) $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$ is zero

If this is the case, A is called a *zero object* of $\mathcal{A}$.

Proof. Straightforward. See for example [Sta21, Lemma 00ZZ]. □

(2.1.3) Proposition. Let A, B be objects of a preadditive category $\mathcal{A}$. If one of $A \times B, A \sqcup B$ exists, then so does the other, and in this case they are isomorphic, via the map

$$A \sqcup B \to A \times B$$

induced by $(\mathrm{id}_A, 0) : A \to A \times B$ and $(0, \mathrm{id}_B) : B \to A \times B$.

We then denote the co/product by $A \oplus B$ and call it the *direct sum*.

Proof. See [Sta21, Lemma 0101]. □

(2.1.4) Remark. An additive functor between preadditive categories preserves direct sums and zero objects.

(2.1.5) Definition. A preadditive category $\mathcal{A}$ is called *additive* if it has all finite direct sums and a zero object.

(2.1.6) Remark. Infinite coproducts and products do not agree. The term *direct sum* in an additive category refers to coproducts in the infinite case.

(2.1.7) Definition. Let $f : A \to B$ be a morphism in a preadditive category $\mathcal{A}$.

(i) A *kernel* of f is a morphism denoted $i : \mathrm{Ker}(f) \to A$ such that $f \circ i = 0$, universal with respect to this property (the notation is justified: by universality, the kernel is unique up to isomorphism).

(ii) A *cokernel* of f is a morphism denoted $p : B \to \mathrm{Coker}(f)$ such that $p \circ f = 0$, universal with respect to this property.

(iii) If f has a kernel, then a *coimage* of f is a cokernel of $\mathrm{Ker}(f) \to A$. It is denoted $A \to \mathrm{Coim}(f)$.

(iv) If f has a cokernel, then an *image* of f is a kernel of $B \to \mathrm{Coker}(f)$. It is denoted $\mathrm{Im}(f) \to B$.

(2.1.8) Remark. It is easy to check that kernels and images are monomorphisms, and that cokernels and coimages are epimorphisms. Using this fact and the universal properties, one deduces a canonical decomposition of $f : A \to B$ as

$$A \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to B$$

(if the coimage and image exist).

(2.1.9) Definition. A category is called *abelian* if it is additive, all kernels and cokernels exist and for every morphism f, the canonical map

$$\mathrm{Coim}(f) \to \mathrm{Im}(f)$$

is an isomorphism.

(2.1.10) Exactness, projectives, and injectives.

- In an abelian category, one can define *injective* and *surjective* morphisms to be those whose kernel (resp. cokernel) is zero. In fact, injective morphisms are precisely the monomorphisms and surjective morphisms are precisely the epimorphisms.

- In an abelian category one defines *complexes* and *exact sequences* in precisely the same way as in a category of modules. The definition of an abelian category contains the exact amount of generality needed to prove results such as the snake lemma, five lemma and long exact sequence in (co)homology. We will discuss complexes in more detail later in this chapter.

- An additive functor F between abelian categories is *right exact, resp. left exact, resp. exact* if for every exact sequence

$$0 \to X \to Y \to Z \to 0,$$

the sequence

$$F(X') \to F(Y) \to F(Z') \to 0,$$

resp.

$$0 \to F(X') \to F(Y) \to F(Z'),$$

resp.

$$0 \to F(X') \to F(Y) \to F(Z') \to 0,$$

is exact, where $(X', Z') = (X, Z)$ or (Z, X) depending on whether F is co- or contravariant. This exactness terminology agrees with the one defined for general functors in (A.1.8). Moreover, left or right exactness in the sense of (A.1.8) implies additivity. See [Sta21, Lemma 010N] for a proof.

- We say that an object P in an abelian category $\mathcal{A}$ is *projective* if the functor $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact, i.e. preserves epimorphisms (surjections). Similarily, an object I is *injective* if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact, i.e. preserves monomorphisms (injections).

□

(2.1.11) Abelian is a structural property. The property of being abelian can be phrased as follows: A category $\mathcal{A}$ is abelian if and only if it satisfies the following properties

(i) There exists a zero object 0 in $\mathcal{A}$. [Recall that a zero object is defined as an object which is both initial and final. If a zero object 0 exists, then there is a unique map $A \to 0$ and a unique map $0 \to B$. The composition of these is denoted $0 \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ and called the *zero morphism*.]

(ii) For all objects A, B in $\mathcal{A}$, their product and coproduct exists, and the map

$$A \sqcup B \to A \times B,$$

defined as $(\mathrm{id}_A, 0)$ on A and $(0, \mathrm{id}_B)$ on B, is an isomorphism.

(iii) Every morphism in $\mathcal{A}$ has a kernel and cokernel.

(iv) The canonical map from the coimage to the image of every morphism is an isomorphism.

Indeed, (i) and (ii) are equivalent to giving an additive structure on $\mathcal{A}$ (see next paragraph), while (iii) and (iv) are the additional conditions required in the definition of an abelian category.

Suppose we have (i) and (ii). Then we have all finite products, so we only need to deduce an abelian group structure on the hom sets. Denote, for all A, B,

$$\varphi_{A,B} : A \times B \to A \sqcup B$$

the inverse to the isomorphism

$$A \sqcup B \to A \times B.$$

Let

$$\Delta : A \to A \times A$$

be the diagonal and let

$$\pi : B \sqcup B \to B$$

be the morphism induced by $(\mathrm{id}_B, \mathrm{id}_B)$. Let $f, g \in \mathrm{Hom}_{\mathcal{A}}(A, B)$. Define

$$f + g : A \to B$$

as the composition

$$A \xrightarrow{\Delta} A \times A \xrightarrow{(f,g)} B \times B \xrightarrow{\varphi_{B,B}} B \sqcup B \xrightarrow{\pi} B$$

This data endows $\mathrm{Hom}_{\mathcal{A}}(A, B)$ with an abelian group structure, where the identity is given by the zero morphism defined above (see [Lur17, Definition 1.1.2.1]).

The converse is given by proposition (2.1.3).

We note that the properties (i)-(iv) are all statements of the type: some limit or colimit exists or has a certain property, or something is a limit or colimit. In particular, no extra structure is defined on the category as the original definition of abelian category might suggest (equipping the hom sets with an abelian group structure). $\square$

2.2 Condensed abelian groups

(2.2.1) Definition. A *condensed abelian group* is a sheaf of abelian groups on one of the sites $\mathrm{ED}, \mathrm{Prof}, \mathrm{CHaus}$. The category of condensed abelian groups is denoted CondAb.

(2.2.2) We use the following characterisation of condensed abelian groups: A condensed abelian group is a presheaf T from ED to abelian groups, taking finite disjoint unions to the corresponding finite products. $\square$

(2.2.3) Theorem. All limits and colimits exist in CondAb. Moreover, they are formed objectwise on extremally disconnected sets.

(2.2.4) Remark. The theorem is saying that the value of the limit or colimit of a functor $I \to \mathrm{CondAb}$, $i \mapsto M_i$, at an extremally disconnected set S, is the limit or colimit in the category of abelian groups of the functor $i \mapsto M_i(S)$. This is *not* necessarily true for any profinite S.

Proof of theorem (2.2.3). Let $I \to \mathrm{CondAb}$, $i \mapsto M_i$ be a functor. We know that $\varinjlim_i M_i$ and $\varprojlim_i M_i$ exist in the presheaf category $\mathrm{Fun}(\mathrm{ED}^{\mathrm{op}}, \mathrm{Ab})$ and are computed objectwise, i.e.

$$(\varinjlim_i M_i)(S) = \varinjlim_i M_i(S)$$
$$(\varprojlim_i M_i)(S) = \varprojlim_i M_i(S)$$

(this is a general categorical fact, see e.g. [nLa21] for a proof). Since limits and colimits commute with finite products in Ab (because a finite product is a finite direct sum in the category of abelian groups, i.e. both a limit and a colimit, and limits commute with limits and colimits commute with colimits in all categories), we obtain

$$
\begin{aligned}
(\varinjlim_i M_i)\left(\coprod_{j=1}^n S_j\right) &= \varinjlim_i M_i\left(\coprod_{j=1}^n S_j\right)\\
&= \varinjlim_i \left(\prod_{j=1}^n M_i(S_j)\right)\\
&= \prod_{j=1}^n \varinjlim_i M_i(S_j)\\
&= \prod_{j=1}^n (\varinjlim_i M_i)(S_j)
\end{aligned}
$$

which means that $\varinjlim_i M_i \in \mathrm{CondAb}$ as desired. The corresponding result for $\varprojlim_i M_i$ is obtained in the exact same way. □

(2.2.5) Corollary. The category of condensed abelian groups is an abelian category satisfying the following additional properties

(AB3) All colimits exist.

(AB3*) All limits exist.

(AB4) Direct sums are exact.

(AB4*) Products are exact.

(AB5) Filtered colimits are exact

(AB6) For any index set J and filtered categories I_j, $j \in J$, with functors $i \mapsto M_i$ from I_j to condensed abelian groups, the natural map

$$\varinjlim_{(i_j \in I_j)_j} \prod_{j \in J} M_{i_j} \to \prod_{j\in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

is an isomorphism.

Proof. These are all results about limits and colimits that are true in the category of abelian groups (by remark (2.1.11), the property of being abelian can be phrased in terms of statements about limits and colimits). By theorem (2.2.3), we deduce them for CondAb. □

(2.2.6) Theorem. The category CondAb is generated by compact projective objects.

(2.2.7) Remark. We recall some terminology:

- A category $\mathcal{C}$ is *generated by* $E \subset \operatorname{Ob} \mathcal{C}$ if for any pair of distinct morphisms $f, g : X \to Y$ there exists an object $U \in E$ and morphism $h : U \to X$ such that $f \circ h \neq g \circ h$.
- An object P is *projective* if $\operatorname{Hom}(P, -)$ preserves epimorphisms.
- An object P is *compact* if $\operatorname{Hom}(P, -)$ commutes with filtered colimits.

Proof of theorem (2.2.6). The forgetful functor from condensed abelian groups to condensed sets preserves limits. Since the categories involved are essentially small, and condensed abelian groups admit all limits, the adjoint functor theorem (theorem (A.2.3)) implies that it has a left adjoint $T \mapsto \mathbb{Z}[T]$ (this is the sheafification of $S \mapsto \mathbb{Z}[T(S)]$). For an extremally disconnected set S and condensed abelian group M, we have

$$\begin{aligned} \operatorname{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[\underline{S}], M) &\simeq \operatorname{Hom}_{\mathrm{CondSet}}(\underline{S}, M) && \text{(adjunction)} \\ &\simeq \operatorname{Hom}_{\mathrm{CondSet}}(\operatorname{Hom}_{\mathrm{ED}}(-, S), M) && \text{(by definition)} \\ &\simeq M(S) && \text{(by Yoneda)} \end{aligned}$$

Since $M \mapsto M(S)$ commutes with all limits and colimits, and preserves epimorphisms, we conclude that $\mathbb{Z}[\underline{S}]$ is compact and projective for every extremally disconnected set S.

Take a condensed abelian group M and consider the set of all extremally disconnected sets S with a map $f : \mathbb{Z}[\underline{S}] \to M$ and consider the induced map

$$h : \bigoplus \mathbb{Z}[\underline{S}] \to M$$

We want to show that this is surjective. Take an extremally disconnected set S and $x \in M(S)$. The corresponding $f \in \operatorname{Hom}(\operatorname{Hom}(-, S), M)$ (via the Yoneda bijection) has x in its image by definition ($x = f(\mathrm{id}_S)$). Therefore h is surjective. If we have $\varphi, \psi : M \to M'$ a pair of distinct morphisms in CondAb, we have $\varphi \circ h \neq \psi \circ h$ since h is an epimorphism. This implies that one of the $f : \mathbb{Z}[\underline{S}] \to M$ separates φ and ψ, as desired. □

(2.2.8) Corollary. The category CondAb has enough projectives.

Proof. These are given by direct sums of the compact projectives $\mathbb{Z}[\underline{S}]$ for extremally disconnected S. □

(2.2.9) CondAb is a Grothendieck category. A *Grothendieck category* (first introduced by Grothendieck in his Tohoku paper [Gro57], see also [KS06, Definition 8.3.24]) is an abelian category $\mathcal{A}$ satisfying (AB3) and (AB5), possessing a generator (meaning a single object generating the whole category).

Taking the direct sum of all $\mathbb{Z}[\underline{S}]$, we see that CondAb has a generator, and thus is a Grothendieck category. As we will see later, Grothendieck categories provide a nice setting for defining derived functors. By [KS06, Theorem 9.2.6], the category CondAb has enough injectives.

(2.2.10) Remark. The characterisation of CondAb as sheaves on the site ED is the essential ingredient in all of the above. The extremally disconnected sets provide the compact projectives that generate CondAb, so it is really the fact that CondAb is generated by compact projectives that makes it such a nice abelian category.

(2.2.11) Remark. The category CondAb as defined in [Sch19b, Appendix to Lecture II] is *not* a Grothendieck category and in fact does not even have enough injectives. Thus (2.2.9) is a result that relies on the smallness condition we impose on our objects.

(2.2.12) Tensor product. Let M, N be two condensed abelian groups. We define the condensed abelian group $M \otimes N$ as the sheafification of the presheaf which takes an extremally disconnected S to the usual tensor product in abelian groups $M(S) \otimes N(S)$. This defines what is called a *symmetric monoidal tensor product* on CondAb. In particular, there are natural isomorphisms

$$M \otimes N \to N \otimes M \text{ and } M \otimes (N \otimes P) \to (M \otimes N) \otimes P,$$

among other properties. The tensor product satisfies the following properties:

- It represents bilinear maps. This means that we have the usual universal property for the tensor product: bilinear maps

  $$M \times N \to P$$

  correspond to morphisms

  $$M \otimes N \to P.$$

  This follows from the corresponding result about abelian groups.

- For T a condensed set, $\mathbb{Z}[T]$ is flat with respect to this tensor product. This follows from the fact that $\mathbb{Z}[T(S)]$ is free and thus flat as an abelian group.

- For condensed sets T_1, T_2, we have $\mathbb{Z}[T_1 \times T_2] = \mathbb{Z}[T_1] \otimes \mathbb{Z}[T_2]$.

(2.2.13) Internal hom. Since the functor $- \otimes A$ (for A a fixed abelian group) commutes with colimits in abelian groups, the presheaf that the tensor product in CondAb is the sheafification of, commutes with colimits. More precisely, the endofunctor of presheaves of abelian groups on the site ED which takes N to the presheaf $S \mapsto N(S) \otimes M(S)$, for M a fixed condensed abelian group, commutes with colimits. This follows from the corresponding fact, stated above, in abelian groups, since colimits are computed objectwise in the presheaf category. Since sheafification is a left adjoint, we conclude that $- \otimes M$ commutes with colimits in CondAb. Thus it has a right adjoint, which we denote $\underline{\mathrm{Hom}}(M, -)$ and call the *internal hom*. It satisfies the adjunction formula

$$\mathrm{Hom}_{\mathrm{CondAb}}(N \otimes M, P) \simeq \mathrm{Hom}_{\mathrm{CondAb}}(N, \underline{\mathrm{Hom}}(M, P))$$

by definition. More concretely, we use Yoneda to conclude that for any extremally disconnected S,

$$\begin{aligned}\underline{\mathrm{Hom}}(M, N)(S) &= \mathrm{Hom}_{\mathrm{CondSet}}(\underline{S}, \underline{\mathrm{Hom}}(M, N)) \\ &= \mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[\underline{S}], \underline{\mathrm{Hom}}(M, N)) \\ &= \mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[\underline{S}] \otimes M, N).\end{aligned}$$

The above shows that the underlying abelian group of the internal hom is the usual hom.

(2.2.14) More on compact projectve generation. The following two results are taken from [Gin05]. In (2.2.7), we defined a compact object of a category $\mathcal{C}$ to be an object P such that $\mathrm{Hom}(-, P)$ commutes with filtered colimits. A characterisation in abelian categories that reminds of the topological term is given as follows:

- An object P in an abelian category $\mathcal{A}$ is compact if and only if for any collection $(M_i)_{i \in I}$ and map

  $$f : P \to \bigoplus_{i \in I} M_i,$$

  there is a finite set $J \subset I$ such that $\mathrm{Im}\, f$ is a subobject of $\bigoplus_{i \in J} M_i$.

The following is a nice result about abelian categories of which CondAb is only a slight generalisation.

- If an abelian category $\mathcal{A}$ is generated by a single compact projective object, then $\mathcal{A}$ is equivalent to the category of left R-modules for a (not necessarily commutative) ring R.

In the case of CondAb, the situation is not quite as nice, but there is only the slight generalisation that our category has a *set* of compact projective generators (or equivalently a single projective generator, given by their direct sum). □

2.3 The derived category

2.3.1 As a category of complexes

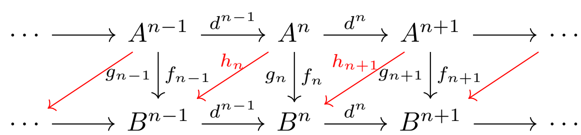
(2.3.1) Definition. Let $\mathcal{A}$ be an abelian category. A *cochain complex* (or simply *complex*) $A^\bullet$ in $\mathcal{A}$ is given by a collection of objects (A^n) indexed by $\mathbb{Z}$ and for each n a morphism $d^n : A^n \to A^{n+1}$, called a *differential*, such that $d \circ d = 0$. A *morphism of complexes* $f : A^\bullet \to B^\bullet$ is a collection of morphisms $(f_n : A^n \to B^n)$ commuting with the differentials ($f \circ d = d \circ f$, omitting the indices). We denote the category of complexes in $\mathcal{A}$ with morphisms of complexes by $\mathrm{CoCh}(\mathcal{A})$.

(2.3.2) Definition. Let $\mathcal{A}$ be an abelian category. Let $f, g : A^\bullet \to B^\bullet$ be two morphisms of complexes in $\mathrm{CoCh}(\mathcal{A})$. A collection h of morphisms

$$h_n : A^n \to B^{n-1}$$

is a *homotopy* from f to g if

$$f - g = d \circ h + h \circ d;$$



The *homotopy category* $K(\mathcal{A})$ is the category whose objects are complexes and whose morphisms are homotopy classes of morphisms of complexes. We let $K^+(\mathcal{A}), K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ denote the full subcategory of $K(\mathcal{A})$ of complexes which are bounded below, bounded above and bounded, respectively.

(2.3.3) Definition. Let $\mathcal{A}$ be an abelian category and $A^\bullet \in \mathrm{CoCh}(\mathcal{A})$. The *$i$-th cohomology group* of $A^\bullet$ is the object

$$H^i(A^\bullet) = \frac{\mathrm{Ker}\, d^i}{\mathrm{Im}\, d^{i-1}}$$

of $\mathcal{A}$ (this quotient of course denotes the cokernel of the map $\mathrm{Im}\, d_{i-1} \to \mathrm{Ker}\, d_i$). For a morphism $f : A^\bullet \to B^\bullet$ we have an induced morphism

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$$

in cohomology. If $H^i(f)$ is an isomorphism, we say that f is a *quasi-isomorphism*.

(2.3.4) Remark. A *chain complex* $A_\bullet$ in $\mathcal{A}$ is defined in the same way as a cochain complex except that the differentials go down in degree:

$$d_n : A_n \to A_{n-1}.$$

The definitions of homotopies, homology groups etc. is analogous. The category of chain complexes is denoted $\mathrm{Ch}(\mathcal{A})$. Usually, all indices are written as subscripts in chain complexes (homological convention) and as superscripts in cochain complexes (cohomological convention). The i-th homology group of a chain complex $A_\bullet$ is thus denoted $H_i(A_\bullet)$. In this chapter we will only discuss cochain complexes.

The following results about the derived category are taken from Gelfand & Manin [GM03] and Huybrechts [Huy06]. Proofs can be found in these references.

(2.3.5) Proposition-definition. Let $\mathcal{A}$ be an abelian category. There exists a category $D(\mathcal{A})$ (unique up to equivalence) called the *derived category of* $\mathcal{A}$ and a functor $Q_\mathcal{A} : \mathrm{CoCh}(\mathcal{A}) \to D(\mathcal{A})$ such that

(i) For any quasi-isomorphism f in $\mathrm{CoCh}(\mathcal{A})$, $Q_\mathcal{A}(f)$ is an isomorphism.

(ii) Any functor $F : \mathrm{CoCh}(\mathcal{A}) \to \mathcal{D}$ transforming quasi-isomorphisms to isomorphisms can be uniquely factorised through $D(\mathcal{A})$ in the following sense: There exists a unique functor $G : D(\mathcal{A}) \to \mathcal{D}$ such that $F = G \circ Q_\mathcal{A}$.

□

(2.3.6) Proposition. The functor $Q_\mathcal{A}$ in proposition-definition (2.3.5) factors through $K(\mathcal{A})$. By abuse of notation, we denote the functor $K(\mathcal{A}) \to D(\mathcal{A})$ by $Q_\mathcal{A}$ as well. □

(2.3.7) Remark. The objects of the derived category $D(\mathcal{A})$ are complexes of objects in $\mathcal{A}$, but if they are quasi-isomorphic in $\mathrm{CoCh}(\mathcal{A})$, then they are isomorphic in $D(\mathcal{A})$. Note that two seemingly very different complexes can be quasi-isomorphic so it is not always helpful to regard the objects of $D(\mathcal{A})$ as complexes. Morphisms in the derived category are nontrivial to describe in general, but sometimes, for nice $\mathcal{A}$, we get a simpler description of $D(\mathcal{A})$.

(2.3.8) Remark. We let $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$ stand for the obvious bounded subcategories.

(2.3.9) Proposition.

- If $\mathcal{A}$ has enough injectives, then every bounded below complex M in $\mathcal{A}$ admits a quasi-isomorphism $M \to I$ where I is a bounded below complex of injectives. Moreover, the category $D^+(\mathcal{A})$ is equivalent to $K^+(\mathcal{I}_\mathcal{A})$, where $\mathcal{I}_\mathcal{A}$ denotes the full subcategory of injective objects of $\mathcal{A}$. The equivalence $K^+(\mathcal{I}_\mathcal{A}) \to D^+(\mathcal{A})$ is given by the functor $Q_\mathcal{A}$

- If $\mathcal{A}$ has enough projectives, then every bounded above complex M in $\mathcal{A}$ admits a quasi-isomorphism $P \to M$ where P is a bounded above complex of projectives. Moreover, the category $D^-(\mathcal{A})$ is equivalent to $K^-(\mathcal{P}_\mathcal{A})$, where $\mathcal{P}_\mathcal{A}$ denotes the full subcategory of projective objects of $\mathcal{A}$. The equivalence $K^-(\mathcal{P}_\mathcal{A}) \to D^-(\mathcal{A})$ is given by the functor $Q_\mathcal{A}$.

□

2.3.2 As a triangulated category

The categories $D^*(\mathcal{A})$ and $K^*(\mathcal{A})$ (where $*$ denotes $+, -, b$ or the empty string) can be equipped with a so-called *triangulated* structure, defined below.

(2.3.10) Definition. Let $\mathcal{C}$ be an additive category and suppose for each $n \in \mathbb{Z}$ given a functor

$$[n] : \mathcal{C} \to \mathcal{C}, \quad X \mapsto X[n],$$

satisfying

$$[n+m] = [n] \circ [m] \text{ and } [0] = \mathrm{id}_\mathcal{C}\,.$$

A *triangle* in $\mathcal{C}$ is a tuple (X, Y, Z, f, g, h) where $f : X \to Y$, $g : Y \to Z$ and $h : Z \to X[1]$. A *morphism of triangles*

$$(a, b, c) : (X, Y, Z, f, g, h) \to (X', Y', Z', f', g', h')$$

is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

(2.3.11) Definition. A *triangulated category* is a triple $(\mathcal{C}, [\;], \mathcal{T})$ where $\mathcal{C}$ is an additive category, $[\;]$ denotes a family of functors as described in definition (2.3.10) and $\mathcal{T}$ is a set of triangles in $\mathcal{C}$ called *distinguished triangles*, satisfying the following four axioms.

(TR1) Any triangle isomorphic to a distinguished triangle is a distinguished triangle. Triangles of the form $(X, X, 0, \mathrm{id}, 0, 0)$ are distinguished. For any morphism $f : X \to Y$ there exists a distinguished triangle (X, Y, Z, f, g, h).

(TR2) The triangle (X, Y, Z, f, g, h) is distinguished if and only if the triangle $(Y, Z, X[1], g, h, -f[1])$ is distinguished.

(TR3) Given distinguished triangles (X, Y, Z, f, g, h) and (X', Y', Z', f', g', h') with morphisms $a : X \to X'$ and $b : Y \to Y'$ such that $b \circ f = f' \circ a$, there exists a morphism $c : Z \to Z'$ such that (a, b, c) is a morphism of triangles. This can be visualised by the following diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \exists c} & & \downarrow{\scriptstyle a[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

(TR4) Given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and distinguished triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$, and (Y, Z, Q_3, g, p_3, d_3), there exist morphisms $a : Q_1 \to Q_2$ and $b : Q_2 \to Q_3$, such that

(i) $(Q_1, Q_2, Q_3, a, b, p_1[1] \circ d_3)$ is a distinguished triangle

(ii) $(\mathrm{id}_X, g, a) : (X, Y, Q_1, f, p_1, d_1) \to (X, Z, Q_2, g \circ f, p_2, d_2)$ is a morphism of triangles

(iii) $(f, \mathrm{id}_Z, b) : (X, Z, Q_2, g \circ f, p_2, d_2) \to (Y, Z, Q_3, g, p_3, d_3)$ is a morphism of triangles

(2.3.12) Definition. A functor $F : \mathcal{C} \to \mathcal{D}$ between triangulated categories is called *triangulated* if

- There is a natural isomorphism of functors $F(-[1]) \to F(-)[1]$.
- Identifying $F(X[1])$ with $F(X)[1]$ via the isomorphism above, F takes distinguished triangles to distinguished triangles, i.e. if

$$(X, Y, Z, f, g, h)$$

is a distinguished triangle of $\mathcal{C}$ then

$$(F(X), F(Y), F(Z), F(f), F(g), F(h))$$

is a distinguished triangle of $\mathcal{D}$.

(2.3.13) Definition. Let $f : A^\bullet \to B^\bullet$ be a morphism of compelexes. We define its *mapping cone* $C(f)$ as the complex given by

$$C(f)^i = A^{i+1} \oplus B^i, \qquad d^i_{C(f)} = \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}.$$

(2.3.14) Definition. Let $\mathcal{A}$ be an abelian category. The *shift functor*

$$-[1] : \mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{A})$$

is defined by

$$A^\bullet[1] = A^{\bullet+1}, \qquad d^i_{A^\bullet[1]} = -d^{i+1}_{A^\bullet}.$$

(2.3.15) Remark. There are natural morphisms of complexes $\tau : B^\bullet \to C(f)$ induced by the injection $B^i \to A^{i+1} \oplus B^i$ and $\pi : C(f) \to A^\bullet[1]$ induced by the projection $A^{i+1} \oplus B^i \to A^{i+1}$.

(2.3.16) Proposition. ([Huy06, Proposition 2.24]). Let $\mathcal{A}$ be an abelian category and $A^\bullet \in \mathrm{CoCh}(\mathcal{A})$. The shift functor $A^\bullet[1] = A^{\bullet+1}$, $d^i_{A^\bullet[1]} = -d^{i+1}_{A^\bullet}$ defines an equivalence of categories $\mathrm{CoCh}(\mathcal{A}) \to \mathrm{CoCh}(\mathcal{A})$, and induces a triangulated structure on $D(\mathcal{A})$ and $K(\mathcal{A})$ (and the bounded versions as well). The distinguished triangles are precisely those isomorphic to triangles of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]$$

□

(2.3.17) Lemma. ([Sta21, Lemma 05QR]). In a triangulated category, let $f : X \to Y$ be a morphism. The following are equivalent

(i) $f : X \to Y$ is an isomorphism

(ii) The triangle $(X, Y, 0, f, 0, 0)$ is distinguished

(iii) For any distinguished triangle (X, Y, Z, f, g, h), we have $Z = 0$.

(2.3.18) Corollary. Let $\mathcal{A}$ be an abelian category. A morphism of complexes $f : X \to Y$ of complexes in $\mathcal{A}$ is a quasi-isomorphism if and only if its mapping cone is acyclic (i.e. quasi-isomorphic to zero).

Proof. Apply lemma (2.3.17) to the triangulated category $D(\mathcal{A})$. □

(2.3.19) Exact sequences and distinguished triangles I. Let $\mathcal{A}$ be an abelian category and

$$0 \to A \to B \to C \to 0$$

an exact sequence in $\mathcal{A}$. Regarding A, B, C as complexes concentrated in degree zero, we get the corresponding elements (still called A, B, C) in $D(\mathcal{A})$. We claim that the exact sequence above corresponds to a distinguished triangle in $D(\mathcal{A})$. We need to show that C is quasi-isomorphic to the mapping cone of $A \to B$. But this is clear, since the mapping cone of $A \to B$ is the two term complex $A \to B$ concentrated in degrees -1 and 0, and its cohomology is C as it sits in the exact sequence with A and B above. We thus have maps

$$\cdots \to C[-1] \to A \to B \to C \to A[1] \to B[1] \to \cdots$$

(2.3.20) Exact sequences and distinguished triangles II. We want to generalise (2.3.19) to exact sequences of complexes. More precisely, we want to prove the following: Let

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

be an exact sequence in CoCh($\mathcal{A}$). Then there is a morphism $C^\bullet \to A^\bullet[1]$ in $D(\mathcal{A})$ such that

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

is a distinguished triangle. To prove this, we show that there is a quasi-isomorphism φ from the mapping cone of $A^\bullet \to B^\bullet$ to $C^\bullet$, and to obtain the desired map $C^\bullet \to A^\bullet[1]$ we take the inverse of φ in the derived category and compose with the natural map from the cone to $A^\bullet[1]$. So denote by f the map $A^\bullet \to B^\bullet$. Define the map $\varphi : C(f) \to C^\bullet$ as

$$\varphi^n : A^{n+1} \oplus B^n \to B^n \to C^n,$$

the composite of the projection and the map $B^\bullet \to C^\bullet$. By definition, φ is an epimorphism.

Now let i be the isomorphism from $A^\bullet$ to $\operatorname{Im} f$. We clearly have a monomorphism $\iota : C(i) \to C(f)$ and it is easy to see that this forms an exact sequence

$$0 \longrightarrow C(i) \xrightarrow{\ \iota\ } C(f) \xrightarrow{\ \varphi\ } C^\bullet \longrightarrow 0$$

Since i is an isomorphism, the cone $C(i)$ is acyclic (by corollary (2.3.18)), and by the long exact sequence in cohomology, we deduce that φ is a quasi-isomorphism, as desired.

2.4 Tensor and hom of complexes

(2.4.1) Double complexes. Consider a commutative diagram of the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{n-1,m-1} & \xrightarrow{d_h} & A^{n,m-1} & \xrightarrow{d_h} & A^{n+1,m-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_v} & & \downarrow{\scriptstyle d_v} & & \downarrow{\scriptstyle d_v} & & \\
\cdots & \longrightarrow & A^{n-1,m} & \xrightarrow{d_h} & A^{n,m} & \xrightarrow{d_h} & A^{n+1,m} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle d_v} & & \downarrow{\scriptstyle d_v} & & \downarrow{\scriptstyle d_v} & & \\
\cdots & \longrightarrow & A^{n-1,m+1} & \xrightarrow{d_h} & A^{n,m+1} & \xrightarrow{d_h} & A^{n+1,m+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

whose entries are objects of an abelian category $\mathcal{A}$, such that $d_h^2 = 0$ and $d_v^2 = 0$. For short, we denote it as $A^{\bullet\bullet}$. We call it a *double complex* of $\mathcal{A}$. We will sometimes denote the vertical differential out of $A^{n,m}$ by $d_v^{n,m}$ and the horizontal one $d_h^{n,m}$ when extra clarity is needed.

(2.4.2) Total complexes. Let $A^{\bullet\bullet}$ be a double complex in an abelian category $\mathcal{A}$.

(i) The *direct sum total complex of* $A^{\bullet\bullet}$ is the comlpex $\mathrm{Tot}^{\oplus}(A^{\bullet\bullet})$ whose n-th term is

$$\mathrm{Tot}^{\oplus}(A^{\bullet\bullet})^n = \bigoplus_{i+j=n} A^{i,j}$$

and whose differential

$$d : \bigoplus_{i+j=n} A^{i,j} \to \bigoplus_{i'+j'=n+1} A^{i',j'}$$

is given by

$$d_{|A^{i,j}} = d_v^{i,j} + (-1)^i d_h^{i,j},$$

where we of course implicitly compose each term with the inclusions

$$X^{i+1,j}, X^{i,j+1} \hookrightarrow \bigoplus_{i'+j'=n+1} A^{i',j'}.$$

(ii) The *product total complex of* $A^{\bullet\bullet}$ is the comlpex $\mathrm{Tot}^{\Pi}(A^{\bullet\bullet})$ whose n-th term is

$$\mathrm{Tot}^{\Pi}(A^{\bullet\bullet})^n = \prod_{i+j=n} A^{i,j}$$

and whose differential

$$d : \prod_{i+j=n-1} A^{i',j'} \to \prod_{i'+j'=n} A^{i,j}$$

is defined by giving its projection to the (i,j) factor:

$$d_v^{i-1,j} + (-1)^i d_h^{i,j-1}$$

where again each term in the sum above is implicitly composed with the relevant projection first.

Note that if for all n, $A^{n-j,j} = 0$ for $|j|$ large enough, then the direct sum total complex and product total complex coincide as the products and direct sums involved are finite.

(2.4.3) Bifunctors. Suppose $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ are abelian categories and

$$F : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$$

a bifunctor which is additive in each variable. For complexes

$$A^{\bullet} \in \mathrm{CoCh}(\mathcal{A}) \text{ and } B^{\bullet} \in \mathrm{CoCh}(\mathcal{A}'),$$

we define a double complex $F(A^{\bullet}, B^{\bullet})$ whose (i,j) entry is $F(A^i, B^j)$ and whose differentials are given by $d_v = F(d_A, \mathrm{id}_B)$ and $d_h = F(\mathrm{id}_A, d_B)$.

(1) If $\mathcal{A}''$ admits countable direct sums, we can define the functor

$$F^\bullet_\oplus : \mathrm{CoCh}(\mathcal{A}) \times \mathrm{CoCh}(\mathcal{A}') \to \mathrm{CoCh}(\mathcal{A}'')$$

by mapping $(A^\bullet, B^\bullet)$ to the direct sum total complex of the double complex defined above, i.e.

$$F^\bullet_\oplus(A^\bullet, B^\bullet) = \mathrm{Tot}^\oplus\left(F(A^\bullet, B^\bullet)\right)$$

(2) If $\mathcal{A}''$ admits countable products, we can define the functor

$$F^\bullet_\Pi : \mathrm{CoCh}(\mathcal{A}) \times \mathrm{CoCh}(\mathcal{A}') \to \mathrm{CoCh}(\mathcal{A}'')$$

by mapping $(A^\bullet, B^\bullet)$ to the product total complex of the double complex defined above, i.e.

$$F^\bullet_\Pi(A^\bullet, B^\bullet) = \mathrm{Tot}^\Pi\left(F(A^\bullet, B^\bullet)\right)$$

(2.4.4) Tensor product of complexes. Suppose $\mathcal{A}$ is an abelian category with symmetric monoidal tensor product. Let $A^\bullet$ and $B^\bullet$ be complexes with entries in $\mathcal{A}$. Define the double complex $C^{\bullet\bullet}$ by $C^{i,j} = A^i \otimes B^j$ with the natural differentials. We define the tensor product of the complexes $A^\bullet$ and $B^\bullet$ as the direct sum total complex of $C^{\bullet\bullet}$.

$$A^\bullet \otimes B^\bullet := \mathrm{Tot}^\oplus(C^{\bullet\bullet}).$$

In other words, if we set $F = - \otimes - : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, we have defined $A^\bullet \otimes B^\bullet$ as $F^\bullet_\oplus(A^\bullet, B^\bullet)$.

(2.4.5) Hom complex. Let $A^\bullet, B^\bullet$ be complexes with entries in the abelian category $\mathcal{A}$. Let $C^{\bullet\bullet}$ be the double complex with $C^{i,j} = \mathrm{Hom}(A^{-i}, B^j)$ and the natural differentials. We define the complex

$$\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Tot}^\Pi(C^{\bullet\bullet}),$$

in other words,

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_{j \in \mathbb{Z}} \mathrm{Hom}(A^j, B^{j+n}).$$

Again, we can put this in the more general framework: if we denote by F the bifunctor $\mathrm{Hom}(-,-) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Ab}$, then we have defined $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = F^\bullet_\Pi(A^\bullet, B^\bullet)$.

(2.4.6) Remark. Of course, the above is still valid if Hom takes values in an abelian category different from Ab, such as $\mathcal{A}$ itself (internal hom) or e.g. a category of modules.

(2.4.7) Proposition. ([KS06, Proposition 11.6.4]). Let $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$ be an additive bifunctor between abelian categories like above. Then if $\mathcal{A}''$ has countable direct sums (resp. countable products) the functor $F^\bullet_\oplus$ (resp. $F^\bullet_\Pi$) induces a well defined bifunctor, triangulated in each variable, between the homotopy categories, $K(\mathcal{A}) \times K(\mathcal{A}') \to K(\mathcal{A}'')$. □

(2.4.8) Tensor-hom adjunction. Suppose $\mathcal{A}$ has an internal $\underline{\mathrm{Hom}}$ and symmetric monoidal tensor product, i.e. an adjunction

$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C))$$

(we have seen that this is the case for $\mathcal{A} = \mathrm{CondAb}$). We can define an internal hom complex $\underline{\mathrm{Hom}}^\bullet(A^\bullet, B^\bullet)$ for complexes by replacing Hom by $\underline{\mathrm{Hom}}$ in (2.4.5). Using the tensor-hom adjunction above, we can show that the adjunction

$$\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet \otimes B^\bullet, C^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, \underline{\mathrm{Hom}}^\bullet(B^\bullet, C^\bullet))$$

still holds in the homotopy category.

2.5 Derived functors

(2.5.1) Definition. Let $\mathcal{A}$ be an abelian category and $F : K^*(\mathcal{A}) \to E$ a triangulated functor (as usual, $K^*(\mathcal{A})$ stands for the unbounded or one of the bounded versions of the homotopy categories).

- A *right derived functor* of F is a triangulated functor $RF : D^*(\mathcal{A}) \to E$ satisfying the universal property given by (i)-(ii) below

  (i) There is a natural transformation

  $$\eta : F \to RF \circ Q_{\mathcal{A}},$$

  (ii) If $G : D^*(\mathcal{A}) \to E$ is a triangulated functor, together with a natural transformation

  $$\eta' : F \to G \circ Q_{\mathcal{A}},$$

  then there is a unique natural transformation $\theta : RF \to G$ such that $\eta' = \theta \circ \eta$.

  As usual with universal properties like this, the pair RF, η is unique up to unique isomorphism.

- A *left derived functor* of F is a triangulated functor $LF : D^*(\mathcal{A}) \to E$ satisfying the universal property given by (i)-(ii) below

  (i) There is a natural transformation

  $$\eta : LF \circ Q_{\mathcal{A}} \to F,$$

(ii) If $G : D^*(\mathcal{A}) \to E$ is a triangulated functor, together with a natural transformation
$$\eta' : G \circ Q_{\mathcal{A}} \to F,$$
then there is a unique natural transformation $\theta : G \to LF$ such that $\eta' = \eta \circ \theta$.

Again, the pair LF, η is unique up to unique isomorphism.

(2.5.2) Remark. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. By abuse of notation, we still denote the induced triangulated functor $K^*(\mathcal{A}) \to K^*(\mathcal{B})$ (given by applying F termwise) by F. This is the most common setting for deriving a functor. In the case that $RF : D^*(\mathcal{A}) \to K^*(\mathcal{B})$ exists, we denote by RF again the composition $Q_{\mathcal{B}} \circ RF$ (after all, the functor $Q_{\mathcal{B}}$ is triangulated as well). Same for LF.

(2.5.3) Definition. Let $F : K^*(\mathcal{A}) \to K^*(\mathcal{B})$ be a triangulated functor which has a right derived functor RF. Then we define the *higher derived functors* of F as
$$R^i F(A^\bullet) = H^i(RF(A^\bullet))$$

(2.5.4) Definition. Let $\mathcal{A}, \mathcal{A}'$ be abelian categories and let
$$F : K^*(\mathcal{A}) \times K^\dagger(\mathcal{A}') \to E$$
be a triangulated bifunctor (i.e. triangulated in each variable). Here, $*$ and $\dagger$ stand for (possibly different) boundedness conditions.

- A *right derived bifunctor* of F is a triangulated bifunctor
$$RF : D^*(\mathcal{A}) \times D^\dagger(\mathcal{A}') \to E$$
together with a natural transformation
$$\eta : F \to RF \circ (Q_{\mathcal{A}} \times Q_{\mathcal{A}'}),$$
satisfying the same universal property as in Defintion (2.5.1). The pair RF, η is unique up to unique isomorphism.

- A *left derived bifunctor* of F is a triangulated bifunctor
$$LF : D^*(\mathcal{A}) \times D^\dagger(\mathcal{A}') \to E$$
together with a natural transformation
$$\eta : LF \circ (Q_{\mathcal{A}} \times Q_{\mathcal{A}'}) \to F,$$
satisfying the same universal property as in Defintion (2.5.1). The pair LF, η is unique up to unique isomorphism.

(2.5.5) Definition. Let $\mathcal{A}$ be an abelian category. Recall that a complex $N \in K(\mathcal{A})$ is *acyclic* if $H^i(N) = 0$ for all i.

(1) A complex $I \in K(\mathcal{A})$ is called *K-injective* if the complex $\mathrm{Hom}^\bullet(N, I)$ is acyclic for every acyclic N.

(2) A *K-injective resolution of* $M \in K(\mathcal{A})$ is a quasi-isomorphism $M \to I$ in $K(\mathcal{A})$ where I is K-injective.

(3) We say that $K(\mathcal{A})$ *has enough K-injectives* if every object has a K-injective resolution.

We also have the dual notion:

(1) A complex $P \in K(\mathcal{A})$ is called *K-projective* if the complex $\mathrm{Hom}^\bullet(P, N)$ is acyclic for every acyclic N.

(2) A *K-projective resolution of* $M \in K(\mathcal{A})$ is a quasi-isomorphism $P \to M$ in $K(\mathcal{A})$ where P is K-projective.

(3) We say that $K(\mathcal{A})$ *has enough K-projectives* if every object has a K-projective resolution.

(2.5.6) Theorem. ([Yek15b, Theorems 4.2 and 4.7]). Let $\mathcal{A}$ be an abelian category.

- Suppose $K(\mathcal{A})$ has enough K-injectives. Then every triangulated functor $F : K(\mathcal{A}) \to E$ has a right derived functor (RF, η). Moreover, if I is K-injective, then the morphism $\eta : F(I) \to RF(Q_{\mathcal{A}}(I))$ is an isomorphism.

- Suppose $K(\mathcal{A})$ has enough K-projectives. Then every triangulated functor $F : K(\mathcal{A}) \to E$ has a left derived functor (LF, η). Moreover, if P is K-projective, then the morphism $\eta : LF(Q_{\mathcal{A}}(I)) \to F(I)$ is an isomorphism.

□

(2.5.7) Theorem. ([Yek15a, Theorem 14.3.2]). Let $\mathcal{A}, \mathcal{A}'$ be abelian categories and $F : K^*(\mathcal{A}) \times K^\dagger(\mathcal{A}') \to E$ a triangulated bifunctor. Suppose there exists a full triangulated subcategory $\mathcal{I}$ of $K^\dagger(\mathcal{A}')$ with the following two properties:

(i) If $\psi : I \to I'$ and $\varphi : M \to M'$ are quasi-isomorphisms in $\mathcal{I}$ and $K^*(\mathcal{A})$ respectively, then

$$F(\varphi, \psi) : F(M, I) \to F(M', I')$$

is an isomorphism in E.

(ii) Every object N of $K^\dagger(\mathcal{A}')$ admits a quasi-isomorphism $N \to I$ with $I \in \mathcal{I}$.

Then F has a right derived bifunctor (RF, η). If $I \in \mathcal{I}$ and $M \in K^*(\mathcal{A})$, then the morphism

$$\eta : F(M, I) \to RF(Q_{\mathcal{A}}(M), Q_{\mathcal{A}'}(I))$$

is an isomorphism. Thus we *calculate the right derived functor by resolving in the second variable with objects of* $\mathcal{I}$ (note that there is nothing special about the second variable, it might as well have been $K^*(\mathcal{A})$ that had a subcategory like $\mathcal{I}$ and then we would have resolved in the first variable). □

(2.5.8) Theorem. (Dual to theorem (2.5.7)). Let $\mathcal{A}, \mathcal{A}'$ be abelian categories and $F : K^*(\mathcal{A}) \times K^\dagger(\mathcal{A}') \to E$ a triangulated bifunctor. Suppose there exists a full triangulated subcategory $\mathcal{P}$ of $K^*(\mathcal{A})$ with the following two properties:

(i) If $\psi : P \to P'$ and $\varphi : N \to N'$ are quasi-isomorphisms in $\mathcal{P}$ and $K^\dagger(\mathcal{A}')$ respectively, then

$$F(\psi, \varphi) : F(P, N) \to F(P', N')$$

is an isomorphism in E.

(ii) Every object M of $K^*(\mathcal{A})$ admits a quasi-isomorphism

$$P \to M \text{ with } P \in \mathcal{P}$$

Then F has a left derived bifunctor (LF, η). If $P \in \mathcal{P}$ and $N \in K^\dagger(\mathcal{A}')$, then the morphism

$$\eta : LF(Q_{\mathcal{A}}(P), Q_{\mathcal{A}'}(N)) \to F(P, N)$$

is an isomorphism. Thus we *calculate the left derived functor by resolving in the first variable with objects of* $\mathcal{P}$. The same remark about the first vs. second variable as in (2.5.7) applies here. □

(2.5.9) Proposition. If P is a bounded above complex of projectives in $\mathcal{A}$ then P is K-projective. Similarily, a bounded below complex I of injectives is K-injective.

Proof. We postpone the proof of this statement to chapter 4 where we define more powerful tools to deal with results of this kind. See (4.2.5). □

(2.5.10) Theorem. ([Spa88, Corollary 3.5]). Let $\mathcal{A}$ be an abelian category with enough projectives satisfying (AB5) (filtered colimits are exact). Then every complex in $K(\mathcal{A})$ has a K-projective resolution. In other words, $K(\mathcal{A})$ has enough K-projectives. In particular, every complex of condensed abelian groups has a K-projective resolution. □

(2.5.11) Remark. Another criterion for the existence of enough K-projectives in the sense of (2.5.10) is given in [KS06, Theorem 14.4.3]: $\mathcal{A}$ needs to have enough projectives and satisfy (AB3) and (AB4). Its dual is given as [KS06, Theorem 14.4.4]: If $\mathcal{A}$ has enough injectives and satisfies (AB3*) and (AB4*), then any complex in $K(\mathcal{A})$ admits a K-injective resolution. In other words, $K(\mathcal{A})$ has enough K-injectives. In conclusion, we have theorem (2.5.12):

(2.5.12) Theorem. The category CondAb has enough K-injectives and K-projectives.

(2.5.13) Remark. It shouould be noted again that the existence of injectives in CondAb relies on the smallness condition, that we are actually working with κ-condensed abelian groups. If one works with the more general category of condensed abelian groups defined in [Sch19b], one loses the existence of enough K-injectives.

(2.5.14) A derived bifunctor as a derived functor. One can easily verify the following: Let

$$F : K^*(\mathcal{A}) \times K^\dagger(\mathcal{A}') \to K(\mathcal{A}'')$$

be a bifunctor for which the right derived bifunctor

$$RF : D^*(\mathcal{A}) \times D^\dagger(\mathcal{A}') \to D(\mathcal{A}'')$$

exists. Then for $M \in K^\dagger(\mathcal{A}')$, the functor

$$RF(-, M) : D^*(\mathcal{A}) \to D(\mathcal{A}'')$$

is the right derived functor of

$$F(-, M) : K^*(\mathcal{A}) \to K(\mathcal{A}'').$$

The analogous statements for left derived functors, as well as keeping the other variable fixed, of course hold as well.

2.6 The derived category of condensed abelian groups

(2.6.1) K-flat complexes. Let $\mathcal{A}$ be a closed symmetric monoidal abelian category with countable direct sums. This means that we have a commutative tensor product which we can extend to complexes, and an internal hom, as described above for CondAb, but other examples of such $\mathcal{A}$ are the category of abelian groups, or more generally the category of all modules over a commutative ring R. A complex $P^\bullet$ is called *K-flat* if for any acyclic complex $M^\bullet$, the tensor product $P^\bullet \otimes M^\bullet$ is acyclic. In Ab and even in R-Mod for any commutative ring R, a K-projective complex is K-flat (see [Yek19, Proposition 10.3.4]). This might also hold in CondAb, but we only prove a bounded version.

(2.6.2) Lemma. Let $P^\bullet$ be a bounded above complex of projective condensed abelian groups. Then $P^\bullet$ is K-flat.

Proof. See chapter 4, when we have defined more powerful tools to deal with results of this kind; (4.2.5). □

(2.6.3) Lemma. Let $\mathcal{A}$ be an abelian category like in (2.6.1). Further, suppose that $K^*(\mathcal{A})$ has enough K-flats, i.e. that every $M \in K^*(\mathcal{A})$ admits a quasi-isomorphism $Q \to M$ where Q is K-flat. If $\varphi : M \to M'$ is a quasi-isomorphism in $K(\mathcal{A})$ and $\psi : P \to P'$ is a quasi-isomorphism of K-flat complexes, then $\varphi \otimes \psi : M \otimes P \to M' \otimes P'$ is a quasi-isomorphism.

Proof. It suffices to prove the statement first in the case $P = P'$, $\psi = \mathrm{id}_P$ and then in the case $M = M'$, $\varphi = \mathrm{id}_M$, since a composition of quasi-isomorphisms is a quasi-isomorphism.

Suppose first that $P = P'$ and $\psi = \mathrm{id}_P$. Let N be the mapping cone of φ. Quasi-isomorphisms in $K(\mathcal{A})$ are precisely those morphisms whose mapping cone is acyclic, see corollary (2.3.18). Since the functor $- \otimes P$ is triangulated, $N \otimes P$ is the mapping cone of $M \otimes P \to M' \otimes P$. And since P is K-flat, $N \otimes P$ is acyclic.

Now suppose that $M = M'$ and $\varphi = \mathrm{id}_M$. Let $Q \to M$ be a K-flat resolution (i.e. a quasi-isomorphism with Q a K-flat complex). Consider the commutative diagram

$$\begin{array}{ccc} Q \otimes P & \longrightarrow & Q \otimes P' \\ \downarrow & & \downarrow \\ M \otimes P & \longrightarrow & M \otimes P' \end{array}$$

The former case shows that the top horizontal arrow and both the vertical arrows are quasi-isomorphisms, and we conclude that the bottom horizontal arrow is as well, which is what we wanted. □

(2.6.4) Derived tensor product. Consider the bifunctor

$$F := - \otimes - : K(\mathrm{CondAb}) \times K(\mathrm{CondAb}) \to K(\mathrm{CondAb})$$

given by the tensor product of complexes as defined in (2.4.4). This bifunctor is triangulated in each variable by (2.4.7). We want to use theorem (2.5.8) to define its left derived functor. In this work we will restrict ourselves to the bounded above case, although the unbounded derived functor does exist. By lemma (2.6.2), the category $K^-(\mathrm{CondAb})$ has enough K-flats (consisting of bounded above complexes of projectives), and thus lemma (2.6.3) allows us to use theorem (2.5.8) to conclude that there exists a *derived tensor product*

$$- \otimes^L - : D^-(\mathrm{CondAb}) \times D^-(\mathrm{CondAb}) \to D(\mathrm{CondAb}).$$

Concretely, for bounded above complexes $M^\bullet$ and $N^\bullet$ of condensed abelian groups, we take a projective resolution $P^\bullet \to M^\bullet$, and we have

$$M^\bullet \otimes^L N^\bullet = P^\bullet \otimes N^\bullet.$$

We get the same result by taking a projective resolution $Q^\bullet \to N^\bullet$:

$$M^\bullet \otimes^L N^\bullet = M^\bullet \otimes Q^\bullet.$$

(2.6.5) Derived hom. A proof nearly identical to that of lemma (2.6.3) shows that if we have a quasi-isomorphism $M' \to M$ of complexes of condensed abelian groups and $I \to I'$ of K-injective complexes of condensed abelian groups, then the induced

$$\operatorname{Hom}^\bullet(M, I) \to \operatorname{Hom}^\bullet(M', I')$$

is a quasi-isomorphism. Thus we can take the K-injectives as the subcategory $\mathcal{I}$ in theorem (2.5.7) to define the *derived hom* of condensed abelian groups. Explicitly, we compute it as follows. For complexes M, N of condensed abelian groups, take a K-injective resolution $N \to I$. Then

$$R\operatorname{Hom}(M, N) = \operatorname{Hom}^\bullet(M, I).$$

Since K-injectives in the opposite category are K-projectives in the original category, we can also take a K-projective resolution $P \to M$ and calculate

$$R\operatorname{Hom}(M, N) = \operatorname{Hom}^\bullet(P, N).$$

(2.6.6) Derived internal hom. Consider the bifunctor

$$\underline{\operatorname{Hom}}^\bullet(-,-) : K^+(\mathrm{CondAb}^{\mathrm{op}}) \times K(\mathrm{CondAb}) \to K(\mathrm{CondAb})$$

defined in the same way as $\operatorname{Hom}^\bullet$ but with internal $\underline{\operatorname{Hom}}$ instead of the regular Hom. We want to use the K-injectives and theorem (2.5.7) to define its right derived functor. To be able to do this, we need to show that for any acyclic (bounded below) complex M of condensed abelian groups and K-injective complex I of condensed abelian groups, the complex

$$\underline{\operatorname{Hom}}^\bullet(M, I)$$

is acyclic, i.e. that for every extremally disconnected S, the complex

$$\operatorname{Hom}(M \otimes \mathbb{Z}[\underline{S}], I)$$

is acyclic. But this is clear, since $\mathbb{Z}[\underline{S}]$ is projective and bounded, thus K-flat on $K^-(\mathrm{CondAb})(= K^+(\mathrm{CondAb}^{\mathrm{op}}))$ (by (2.6.2)), and therefore $M \otimes \mathbb{Z}[\underline{S}]$ is acyclic, and since I is K-injective, we conclude that

$$\operatorname{Hom}(M \otimes \mathbb{Z}[\underline{S}], I)$$

is acyclic. Now we use the usual argument to conclude that we can calculate $R\underline{\operatorname{Hom}}$: for complexes M, N of condensed abelian groups (with M bounded below) we take a K-injective resolution $N \to I$ and we have

$$R\underline{\operatorname{Hom}}(M, N) = \underline{\operatorname{Hom}}^\bullet(M, I).$$

By definition, $R\operatorname{Hom}$ is obtained by evaluating $R\underline{\operatorname{Hom}}$ at the point.

(2.6.7) Tensor-hom adjunction. Let L, M, N be complexes of condensed abelian groups (satisfying the relevant boundedness conditions: L, M bounded above). Take a projective resolution $P \to L$ and a K-injective resolution $N \to I$. Then by (2.4.8) we have the adjunction

$$\begin{aligned}\operatorname{Hom}_{D(\mathrm{CondAb})}(L \otimes^L M, N) &= \operatorname{Hom}_{K(\mathrm{CondAb})}(P \otimes M, I)\\ &= \operatorname{Hom}_{K(\mathrm{CondAb})}(P, \underline{\operatorname{Hom}}^\bullet(M, I))\\ &= \operatorname{Hom}_{D(\mathrm{CondAb})}(L, \underline{\operatorname{Hom}}^\bullet(M, I))\\ &= \operatorname{Hom}_{D(\mathrm{CondAb})}(L, R\underline{\operatorname{Hom}}(M, N)).\end{aligned}$$

(2.6.8) Derived tensor-hom adjunction. We have the adjunction

$$\operatorname{Hom}_{D(\mathrm{CondAb})}(M \otimes^L N, P) = \operatorname{Hom}_{D(\mathrm{CondAb})}(M, R\underline{\operatorname{Hom}}(N, P)).$$

We want to show the derived version

$$R\operatorname{Hom}(M \otimes^L N, P) = R\operatorname{Hom}(M, R\underline{\operatorname{Hom}}(N, P)).$$

Evaluating on the point, it suffices to show that

$$R\underline{\operatorname{Hom}}(M \otimes^L N, P) = R\underline{\operatorname{Hom}}(M, R\underline{\operatorname{Hom}}(N, P)).$$

This follows formally from the fact that they represent the same functor: for any $X \in D(\mathrm{CondAb})$,

$$\begin{aligned}&\operatorname{Hom}_{D(\mathrm{CondAb})}\left(X, R\underline{\operatorname{Hom}}(M \otimes^L N, P)\right)\\ &= \operatorname{Hom}_{D(\mathrm{CondAb})}\left(X \otimes^L M \otimes^L N, P\right)\\ &= \operatorname{Hom}_{D(\mathrm{CondAb})}\left(X \otimes^L M, R\underline{\operatorname{Hom}}(N, P)\right)\\ &= \operatorname{Hom}_{D(\mathrm{CondAb})}\left(X, R\underline{\operatorname{Hom}}(M, R\underline{\operatorname{Hom}}(N, P))\right).\end{aligned}$$

This means that we can calculate the S-valued points of $R\underline{\operatorname{Hom}}$ as follows

$$R\underline{\operatorname{Hom}}(M, N)(S) = R\operatorname{Hom}(M \otimes \mathbb{Z}[\underline{S}], N).$$

2.7 Derived limits and colimits.

(2.7.1) Definition. Let $\mathcal{D}$ be a triangulated category and let (K_n, f_n) be an inverse system of objects in $\mathcal{D}$ indexed by $\mathbb{N}$ (i.e. for all n, f_{n+1} is a morphism $K_{n+1} \to K_n$. An object K is a *derived limit* or *homotopy limit* of the system, if the product $\prod_{n\in\mathbb{N}} K_n$ exists and there is a distinguished triangle

$$K \to \prod_{n\in\mathbb{N}} K_n \to \prod_{n\in\mathbb{N}} K_n \to K[1]$$

where the map between the products is given as follows. For each $n \in \mathbb{N}$, we define $K_{n+1} \to \prod_{n\in\mathbb{N}} K_n$ by $\mathrm{id}_{K_{n+1}}$ to the $(n+1)$-st factor, $-f_n$ to the n-th factor, and 0 otherwise.

(2.7.2) Remark. Using the axioms of a triangulated category one can deduce that a derived limit is unique up to isomorphism and that it exists whenever the product $\prod_{n\in\mathbb{N}} K_n$ exists. We write

$$K = R\varprojlim_n K_n.$$

(2.7.3) Definition. Let $\mathcal{D}$ be a triangulated category and let (K_n, f_n) be a system of objects in $\mathcal{D}$ indexed by $\mathbb{N}$ (i.e. for all n, f_n is a morphism $K_n \to K_{n+1}$. An object K is a *derived colimit* or *homotopy colimit* of the system, if the direct sum $\bigoplus_{n\in\mathbb{N}}$ exists and there is a distinguished triangle

$$\bigoplus_{n\in\mathbb{N}} K_n \to \bigoplus_{n\in\mathbb{N}} K_n \to K \to \bigoplus_{n\in\mathbb{N}} K_n[1]$$

where the map between the direct sums is given as follows. For each $n \in \mathbb{N}$, we define $K_n \to \bigoplus_{n\in\mathbb{Z}} K_n$ by $\mathrm{id}_{K_n} - f_n$.

(2.7.4) Remark. The same remark about existence and uniqueness applies to derived colimits. We denote

$$K = \mathrm{hocolim}_n\, K_n.$$

(2.7.5) Derived limits and colimits in $D(\mathcal{A})$. Let $\mathcal{A}$ be an abelian category with exact countable products and direct sums (for example, $\mathcal{A} = \mathrm{CondAb}$). Then $D(\mathcal{A})$ has countable products and direct sums. Moreover, if we have a countable collection $(K_n^\bullet)_n$ of objects of $D(\mathcal{A})$ represented by complexes as the notation suggests, then the product and direct sum are obtained by taking the termwise product or direct sum (see [Sta21, Lemma 0A5L] and [Sta21, Lemma 07KC]).

We end this section with two lemmas which will be useful in chapter 4.

(2.7.6) Lemma. Let $\mathcal{A}$ be an abelian category with countable exact products and let X be an object of $D(\mathcal{A})$. Then X is a derived limit of the constant inverse system $(X, \mathrm{id}_X)_{n\in\mathbb{N}}$.

Proof. Denote also by X a complex representing X. We have an exact sequence of complexes

$$0 \to X \to \prod_{n\in\mathbb{N}} X \to \prod_{n\in\mathbb{N}} X \to 0$$

where the map between the products is given by $(\dots, 0, \mathrm{id}, -\mathrm{id}, 0, \dots)$ on each factor and thus by (2.3.20) the distinguished triangle in $D(\mathcal{A})$ required for the derived limit. □

(2.7.7) Lemma. Let $\mathcal{A}$ be an abelian category in which colimits over the filtered category $\mathbb{N} = \{0 \to 1 \to 2 \to \cdots\}$ exist and are exact (such as $\mathcal{A} = \mathrm{CondAb}$). Let $(L_n^\bullet)$ be a system in $\mathrm{CoCh}(\mathcal{A})$. Then the complex obtained by taking the termwise colimit of these complexes is a homotopy colimit in $D(\mathcal{A})$.

Proof. See [Sta21, Lemma 0949]. □

Chapter 3

Condensed cohomology of compact Hausdorff spaces

In this chapter, we introduce a cohomology theory of condensed abelian groups, which we apply to compact Hausdorff spaces and relate it to classical notions of cohomology on such spaces. section 3.1 is devoted to introducing the simplicial methods needed to give an explicit description of the cohomology then defined in section 3.2.

3.1 Simplicial objects and hypercovers

3.1.1 Basic definitions and constructions

(3.1.1) Definition. Let Δ denote the category of finite, nonempty ordinals

$$[n] = (0 < \cdots < n)$$

and increasing maps between them. Let $\mathcal{C}$ be a category.

- A *simplicial object of* $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. The category of simplicial objects of $\mathcal{C}$ is denoted $\mathrm{Simp}(\mathcal{C})$.
- A *cosimplicial object of* $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. The category of cosimplicial objects of $\mathcal{C}$ is denoted $\mathrm{CoSimp}(\mathcal{C})$.

(3.1.2) Remark.

- A cosimplicial object of $\mathcal{C}$ can equivalently be defined as a simplicial object of $\mathcal{C}^{\mathrm{op}}$.
- If $\mathcal{C}$ is the category of sets, finite sets, abelian groups, etc. then the objects of $\mathrm{Simp}(\mathcal{C})$ are called simplicial sets, simplicial finite sets, simplicial

abelian groups, etc., respectively. The same terminology is used for cosimplicial objects.

(3.1.3) Definition. We denote by $\Delta[n]$ is the simplicial set represented by $[n]$, i.e. for all k,

$$\Delta[n]([k]) = \mathrm{Hom}_{\Delta}([k],[n]).$$

(3.1.4) Proposition. (See [Val20]). A simplicial object of a category $\mathcal{C}$ is given by precisely the following data

- A collection (X_n) of objects of $\mathcal{C}$ indexed by the natural numbers
- For each $0 \leq i \leq n$, a morphism $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ (*faces* and *degeneracies* respectively)

satisfying the *simplicial identities*

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \\ s_i s_j &= s_{j+1} s_i \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{for } i < j \\ \mathrm{id} & \text{for } i = j \text{ and } i = j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases} \end{aligned}$$

□

(3.1.5) Remark. If S is a simplicial set, then we talk about the elements of S_n as the *n-simplices of S*. An n-simplex is then called *degenerate* if it is in the image of a degeneracy (and of course, *nondegenerate* otherwise). proposition (3.1.4) implies that the nondegenerate simplices at each level determine the simplicial set completely (all degenerate simplices are obtained by applying a series of degeneracies to a nondegenerate simplex at a lower level).

(3.1.6) Remark. There is of course an analogue of proposition (3.1.4) for cosimplicial objects and the analogue of the remark above holds as well. A cosimplicial object X has *cofaces* and *codegeneracies*

$$\delta_i : X_{n-1} \to X_n, \quad \sigma_i : X_{n+1} \to X_n,$$

for $i = 0, \ldots, n$.

(3.1.7) Definition. Let U, V be simplicial sets. We build from these two natural simplicial sets:

- The *product* $U \times V$ has components $(U \times V)_n = U_n \times V_n$, degeneracies and faces being the obvious maps.

- The *internal hom* $\mathrm{Hom}(-,-)$ is defined as the right adjoint to the product above:

$$\mathrm{Hom}_{\mathrm{Simp(Set)}}(U \times V, W) = \mathrm{Hom}_{\mathrm{Simp(Set)}}(U, \mathrm{Hom}(V, W))$$

functorially in U, V, W. Explicitly, one has by the Yoneda Lemma:

$$\begin{aligned}\mathrm{Hom}(U,V)_n &= \mathrm{Hom}_{\mathrm{Simp(Set)}}(\Delta[n], \mathrm{Hom}(U,V)) \\ &= \mathrm{Hom}_{\mathrm{Simp(Set)}}(\Delta[n] \times U, V).\end{aligned}$$

(3.1.8) Definition. Let $\mathcal{C}$ be a category, let U be a simplicial object of $\mathcal{C}$ and let V be a simplicial set. The *product* $U \times V$ (if it exists) is defined as the simplicial object of $\mathcal{C}$ having terms

$$(U \times V)_n = \coprod_{u \in U_n} V_n.$$

For a map $\varphi : [m] \to [n]$, we define

$$(U \times V)(\varphi) : \coprod_{u \in U_n} V_n \to \coprod_{u' \in U_m} V_m$$

to be the map taking the component V_n corresponding to u to the component V_m corresponding to $u' = U(\varphi)(u)$ via the map $V(\varphi)$.

(3.1.9) Remark. It is clear that if we let U in (3.1.8) be a simplicial set, then we retrieve the notion of product of simplicial sets in (3.1.7).

(3.1.10) Definition.
- Let U be a simplicial set and V a cosimplicial set. We define the cosimplicial set $\mathrm{Hom}(U,V)$ as follows:
  - $(\mathrm{Hom}(U,V))_n = \mathrm{Hom}_{\mathrm{Set}}(U_n, V_n)$
  - For $\varphi : [m] \to [n]$ the map $\mathrm{Hom}(U,V)_m \to \mathrm{Hom}(U,V)_n$ is given by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.
- Now let U be a simplicial set, $\mathcal{C}$ a category such that all products appearing below exist, and V a cosimplicial object of $\mathcal{C}$. Inspired by the previous point, we define the cosimplicial object $\mathrm{Hom}(U,V)$ of $\mathcal{C}$ as follows.
  - For each n, we set

$$\mathrm{Hom}(U,V)_n = \prod_{u \in U_n} V_n,$$

i.e. the object representing the functor

$$\mathrm{Hom}_{\mathcal{C}}(-, \mathrm{Hom}(U,V)_n) = \mathrm{Hom}_{\mathrm{Set}}(U_n, \mathrm{Hom}_{\mathcal{C}}(-, V_n))$$

– For $\varphi : [m] \to [n]$, we define a natural transformation between the functors represented by $\mathrm{Hom}(U, V)_m$ and $\mathrm{Hom}(U, V)_n$. These are in bijection (by Yoneda) with morphisms $\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$ in $\mathcal{C}$. For X an object of $\mathcal{C}$, we thus define a morphism

$$\mathrm{Hom}_{\mathrm{Set}}(U_m, \mathrm{Hom}_{\mathcal{C}}(X, V_m)) \to \mathrm{Hom}_{\mathrm{Set}}(U_n, \mathrm{Hom}_{\mathcal{C}}(X, V_n))$$

by $f \mapsto V(\varphi) \circ f \circ U(\varphi)$.

(3.1.11) Remark. Let's spell out the actual morphism

$$\mathrm{Hom}(U, V)_m \to \mathrm{Hom}(U, V)_n$$

given by the natural transformation above. The bijection

$$\begin{aligned}&\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, \mathrm{Hom}(U, V)_m), \mathrm{Hom}_{\mathcal{C}}(-, \mathrm{Hom}(U, V)_n))\\ &\to \mathrm{Hom}_{\mathcal{C}}(\mathrm{Hom}(U, V)_m, \mathrm{Hom}(U, V)_n)\end{aligned}$$

is

$$\eta \mapsto \eta(\mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(\mathrm{Hom}(U,V)_m, \mathrm{Hom}(U,V)_m)}).$$

Now

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{Hom}(U, V)_m, \mathrm{Hom}(U, V)_m) = \mathrm{Hom}_{\mathrm{Set}}\left(U_m, \mathrm{Hom}_{\mathcal{C}}\left(\prod_{u \in U_m} V_m, V_m\right)\right)$$

and the identity corresponds to $u \mapsto p_u$ where p_u denotes the u-th projection. The natural transformation we gave in the definition thus gives the morphism

$$\prod_{u \in U_m} V_m \to \prod_{u' \in U_n} V_n$$

whose projection onto the factor V_n corresponding to $u' \in U_n$ is $V(\varphi) \circ p_{U(\varphi)(u')}$.

(3.1.12) Lemma. Let X be a set which we also regard as a constant simplicial set ($X_n = X$ for all n, maps are all identities). Let $k \in \mathbb{N}$. Let V be any simplicial set. There is a natural bijection

$$\mathrm{Hom}_{\mathrm{Simp(Set)}}(X \times \Delta[k], V) \to \mathrm{Hom}_{\mathrm{Set}}(X, V_k)$$

given by $\gamma \mapsto (\gamma_k)|_{X \times \{\mathrm{id}_{[k]}\}}$

Proof. A morphism $\gamma : X \times \Delta[k] \to V$ is given by a collection of morphisms $(\gamma_n : X \times \mathrm{Hom}_{\Delta}([n], [k]) \to V_n)_{n \in \mathbb{N}}$ such that for any $\varphi : [m] \to [n]$, the diagram

$$\begin{array}{ccc} X \times \mathrm{Hom}_{\Delta}([n], [k]) & \xrightarrow[\gamma_n]{} & V_n \\ {\scriptstyle (\mathrm{id}_X, \circ\varphi)}\downarrow & & \downarrow{\scriptstyle V(\varphi)} \\ X \times \mathrm{Hom}_{\Delta}([m], [k]) & \xrightarrow{\gamma_m} & V_m \end{array}$$

commutes. For any $\alpha \in \mathrm{Hom}_\Delta([n],[k])$, we have

$$V(\varphi) \circ \gamma_n(x, \alpha) = \gamma_m(x, \alpha \circ \varphi).$$

Taking $\alpha = \mathrm{id}_[k]$, we obtain

$$\gamma_m(x, \varphi) = V(\varphi) \circ \gamma_k(x, \mathrm{id}_{[k]}).$$

Thus γ is determined by $(\gamma_k)|_{X \times \{\mathrm{id}_{[k]}\}}$ as desired. Conversely, starting with a map $f : X \to V_k$, we construct γ by setting $\gamma_m(x, \varphi) = V(\varphi)(f(x))$ □

3.1.2 Skeleton functors and its adjoints

(3.1.13) Definition. Denote by $\Delta_{\leq n}$ the full subcategory of Δ consisting of the objects $[0], \dots, [n]$. An *n-truncated simplicial object* is a functor $\Delta^{op}_{\leq n} \to \mathcal{C}$. The category of n-truncated simplicial objects of $\mathcal{C}$ is denoted $\mathrm{Simp}_n(\mathcal{C})$.

- Given a simplicial object U of $\mathcal{C}$ we define $\mathrm{sk}_n(U)$ to be the restriction of the functor U to the subcategory $\Delta_{\leq n}$ This defines the so called *skeleton functor*

  $$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \to \mathrm{Simp}_n(\mathcal{C})$$

- A *coskeleton functor* is a functor

  $$\mathrm{cosk}_n : \mathrm{Simp}_n(\mathcal{C}) \to \mathrm{Simp}(\mathcal{C})$$

  which is right adjoint to the skeleton functor, i.e. there is an isomorphism, functorial in U and V

  $$\mathrm{Hom}_{\mathrm{Simp}(\mathcal{C})}(U, \mathrm{cosk}_n V) \simeq \mathrm{Hom}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$$

- A left adjoint to the skeleton functor is denoted

  $$i_{n!} : \mathrm{Simp}_n(\mathcal{C}) \to \mathrm{Simp}(\mathcal{C}).$$

  The adjunction formula is

  $$\mathrm{Hom}_{\mathrm{Simp}_n(\mathcal{C})}(U, \mathrm{sk}_n V) \simeq \mathrm{Hom}_{\mathrm{Simp}(\mathcal{C})}(i_{n!}U, V)$$

  functorially in U and V.

(3.1.14) Remark. Consider the category $(\Delta/[m])_n$ with objects $[k] \to [m]$ with $k \leq n$. Given an n-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U\{m\} : (\Delta/[m])_n)^{\mathrm{op}} \to \mathcal{C}$$

by

$$U\{m\}([k] \to [m]) = U_k$$

$$U\{m\}\left(\begin{array}{ccc} [k'] & \xrightarrow{\psi} & [k] \\ & \searrow \quad \swarrow & \\ & [m] & \end{array}\right) = U(\psi) : U_k \to U_{k'}$$

Given a morphism $\varphi : [m] \to [m']$ we have a functor $\overline{\varphi} : (\Delta/[m])_n \to (\Delta/[m'])_n$ which maps $\alpha : [k] \to [m]$ to $\varphi \circ \alpha : [k] \to [m']$. We have $U\{m'\} \circ \overline{\varphi} = U\{m\}$.

(3.1.15) Proposition. ([Sta21, Lemma 0183]) If $\mathcal{C}$ has all finite limits, then cosk_n exists for all n and is given by the following formula: For U an n-truncated simplicial object,

$$(\operatorname{cosk}_n U)_m = \varprojlim_{(\Delta/[m])_n^{\mathrm{op}}} U\{m\}$$

and for a map $\varphi : [m] \to [m']$, there is a natural morphism $\operatorname{cosk}_n(\varphi)$ by [Sta21, Lemma 002L]. □

(3.1.16) Remark. Dualizing remark (3.1.14), consider the category $([m]/\Delta)_n$ of objects *under* $[m]$ $[m] \to [k]$ with $k \leq n$. We define a functor

$$U\{m\} : ([m]/\Delta)_n^{\mathrm{op}} \to \mathcal{C}$$

in an analogous way to the previous remark.

(3.1.17) Proposition. ([Sta21, Lemma 018L]). If $\mathcal{C}$ has all finite colimits, then $i_{n!}$ exists for all n and is given by the following formula: For U an n-truncated simplicial object,

$$(i_{n!}U)_m = \varinjlim_{([m]/\Delta)_n^{\mathrm{op}}} U\{m\}$$

and for a map $\varphi : [m] \to [m']$, there is a natural morphism $i_{n!}(\varphi)$ by [Sta21, Lemma 002K]. □

(3.1.18) Remark. In particular, both the adjoints to the skeleton exist in simplicial sets.

(3.1.19) Theorem. Let V be a simplicial set. Then

$$(\operatorname{cosk}_n \operatorname{sk}_n V)_{n+1} = \operatorname{Hom}_{\mathrm{Simp(Set)}}(i_{n!} \operatorname{sk}_n \Delta[n+1], V)$$

Proof. We will show that the sets in the equation represent the same endofunctor on the category of sets. Let X be a set. The functor represented by the left hand side takes X to

$$\begin{aligned} \operatorname{Hom}_{\mathrm{Set}}(X, (\operatorname{cosk}_n \operatorname{sk}_n V)_{n+1}) &= \operatorname{Hom}_{\mathrm{Simp(Set)}}(X \times \Delta[n+1], \operatorname{cosk}_n \operatorname{sk}_n V) \\ &= \operatorname{Hom}_{\mathrm{Simp(Set)}}(\operatorname{sk}_n(X \times \Delta[n+1]), \operatorname{sk}_n V) \\ &= \operatorname{Hom}_{\mathrm{Simp(Set)}}(i_{n!} \operatorname{sk}_n(X \times \Delta[n+1]), V) \end{aligned}$$

where the last two inequalities are the adjunctions and the first one is lemma (3.1.12). The functor represented by the right hand side takes X to

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Set}}(X, \mathrm{Hom}_{\mathrm{Simp(Set)}}(i_{n!}\,\mathrm{sk}_n\,\Delta[n+1], V)\\
&= \mathrm{Hom}_{\mathrm{Set}}(X, \mathrm{Hom}(i_{n!}\,\mathrm{sk}_n\,\Delta[n+1], V)_0)\\
&= \mathrm{Hom}_{\mathrm{Simp(Set)}}(X \times \Delta[0], \mathrm{Hom}(i_{n!}\,\mathrm{sk}_n\,\Delta[n+1], V))\\
&= \mathrm{Hom}_{\mathrm{Simp(Set)}}(X \times i_{n!}\,\mathrm{sk}_n\,\Delta[n+1], V)
\end{aligned}$$

where the first equality follows from the definition of the internal hom, the second is lemma (3.1.12) and the last is the product-hom adjunction plus the obvious fact that $X \times \Delta[0] = X$.

To finish the proof, we thus need to show that

$$i_{n!}\,\mathrm{sk}_n(X \times \Delta[n+1]) = X \times i_{n!}\,\mathrm{sk}_n\,\Delta[n+1].$$

Clearly $\mathrm{sk}_n(X \times \Delta[n+1]) = X \times \mathrm{sk}_n\,\Delta[n]$. To show that we can also move the $i_{n!}$ over the equality sign, we note that for any simplicial set W, the internal hom simplicial set $\mathrm{Hom}(X, W)$ is given by (the set X is here regarded as a constant simplicial set) $\mathrm{Hom}(X, W)_n = \mathrm{Hom}_{\mathrm{Set}}(X, W_n)$ (this follows from lemma (3.1.12)). Hence $(\mathrm{sk}_n\,\mathrm{Hom}(X, W))_m = \mathrm{Hom}_{\mathrm{Set}}(X, (\mathrm{sk}_n\,W)_m)$. Thus if W is a (possibly truncated) simplicial set, we can define a (possibly truncated) simplicial set, still denoted $\mathrm{Hom}(X, W)$, whose n-th component is $\mathrm{Hom}_{\mathrm{Set}}(X, W_n)$, which still satisfies the product-hom adjunction. We obtain

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{Simp(Set)}}(X \times i_{n!}\,\mathrm{sk}_n\,\Delta[n+1], W)\\
&= \mathrm{Hom}_{\mathrm{Simp(Set)}}(i_{n!}\,\mathrm{sk}_n\,\Delta[n+1], \mathrm{Hom}(X, W))\\
&= \mathrm{Hom}_{\mathrm{Simp}_n\mathrm{(Set)}}(\mathrm{sk}_n\,\Delta[n+1], \mathrm{sk}_n\,\mathrm{Hom}(X, W))\\
&= \mathrm{Hom}_{\mathrm{Simp}_n\mathrm{(Set)}}(X \times \mathrm{sk}_n\,\Delta[n+1], \mathrm{sk}_n\,W)\\
&= \mathrm{Hom}_{\mathrm{Simp(Set)}}(i_{n!}(X \times \mathrm{sk}_n\,\Delta[n+1]), W)
\end{aligned}$$

We conclude that

$$X \times i_{n!}\,\mathrm{sk}_n\,\Delta[n+1] = i_{n!}(X \times \mathrm{sk}_n\,\Delta[n+1])$$

and since the right hand side has already been observed to be equal to

$$i_{n!}\,\mathrm{sk}_n(X \times \Delta[n+1],$$

we are done. □

3.1.3 Hypercovers

(3.1.20) Definition. Let $\mathcal{C}$ be a category and X an object of $\mathcal{C}$.

- The *category of semi-representables of* $\mathcal{C}$ is denoted $\mathrm{SR}(\mathcal{C})$. Its objects are families $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$. A morphism $\{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$ is given by a map of sets $\alpha : I \to J$ and for each $i \in I$ a morphism $\varphi_i : X_i \to Y_{\alpha(i)}$ in $\mathcal{C}$.

- We define the *category of semi-representables over* X as

$$\mathrm{SR}(\mathcal{C}, X) := \mathrm{SR}(\mathcal{C}/X).$$

  In other words, its objects are families $\{X_i \to X\}_{i\in I}$ of morphisms with fixed target X, and a morphism

$$\{X_i \to X\}_{i\in I} \to \{Y_j \to X\}_{j\in J}$$

  is given by a map $\alpha : I \to J$ and for each i a map $\varphi_i : X_i \to Y_{\alpha(i)}$ over X (meaning that the natural diagram commutes). There is a natural forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$.

(3.1.21) Finite limits in the category of semirepresentables. If $\mathcal{C}$ is a category with fibre products then $\mathrm{SR}(\mathcal{C})$ has fibre products. Further, if X is an object of $\mathcal{C}$, then $\mathrm{SR}(\mathcal{C}, X)$ has all finite limits. In particular, all coskeleton functors exist in $\mathrm{Simp}(\mathrm{SR}(\mathcal{C}, X))$.

For a proof of this fact, see [Sta21, Lemma 01G2].

But how do the finite limits, and in particular coskeletons, in $\mathrm{SR}(\mathcal{C}, X)$ actually look? Consider first fibre products in $\mathrm{SR}(\mathcal{C})$. Let

$$(\alpha, (f_i)_{i\in I}) : \{U_i\}_{i\in I} \to \{V_j\}_{j\in J}$$

and

$$(\beta, (g_k)_{k\in K}) : \{W_k\}_{k\in K} \to \{V_j\}_{j\in J}$$

be morphisms in $\mathrm{SR}(\mathcal{C})$. Then one can show that the fibre product of these two morphisms is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k\}_{(i,j)\in I\times_{\alpha,J,\beta} K}$$

(where j denotes $\alpha(i) = \beta(k)$).

Since $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$ we can apply the above argument to $\mathcal{C}/X$ to conclude that $\mathrm{SR}(\mathcal{C}, X)$ has fibre products. Namely, the fibre product of

$$(\alpha, (f_i)_{i\in I}) : \{U_i \to X\}_{i\in I} \to \{V_j \to X\}_{j\in J}$$

and

$$(\beta, (g_k)_{k\in K}) : \{W_k \to X\}_{k\in K} \to \{V_j\}_{j\in J}$$

is given by

$$\{U_i \times_{f_i, V_j, g_k} W_k \to X\}_{(i,j)\in I\times_{\alpha,J,\beta} K}.$$

Further, $\mathcal{C}/X$ has the final object $\mathrm{id}_X : X \to X$ and thus $\{\mathrm{id}_X : X \to X\}$ is the final object of $\mathrm{SR}(\mathcal{C}, X)$. Having fibre products and a final object, is equivalent to having all finite limits (see [Sta21, Lemma 002O]) and in fact all finite limits are of the form $A_1 \times_{B_1} A_2 \times_{B_2} \cdots \times_{B_{n-1}} A_n$ (see the previously cited lemma and [Sta21, Lemma 002N]). The above description of fibre products in $\mathrm{SR}(\mathcal{C})$ and $\mathrm{SR}(\mathcal{C}, X)$ still holds for this sort of iterated fibre product of n objects and $2n$ morphisms (writing it out would be painful, and not very helpful). The point is that we can reduce a commutation result below (see (3.1.36)) to the distribution of fibre product over disjoint union.

(3.1.22) Definition. Let $\mathcal{C}$ be a category, with fibre products, equipped with a Grothendieck pretopology $\mathcal{P}$, and let X and object of $\mathcal{C}$. A *hypercover* of X is a simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ such that

$$K_0 := \{X_i \to X\}_{i \in I} \in \mathrm{Cov}_{\mathcal{P}}(X),$$

and for all $n \geq 0$, each component of the canonical map

$$K_{n+1} \to (\mathrm{cosk}_n \, \mathrm{sk}_n \, K)_{n+1}$$

gives a covering family in the following sense: Suppose

$$K_{n+1} = \{X_i \to X\}_{i \in I}$$

and

$$(\mathrm{cosk}_n \, \mathrm{sk}_n \, K)_{n+1} = \{Y_j \to X\}_{j \in J}$$

and that the map is given by $\alpha : I \to J$. Then for each $j \in J$, we have $\{X_i \to Y_j\}_{\alpha(i)=j} \in \mathrm{Cov}_{\mathcal{P}}(Y_j)$.

(3.1.23) Definition. Let $\mathcal{A}$ be an abelian category and U a (co)simplicial object of $\mathcal{A}$. Then the (co)chain complex with the same components as U and the differential given by alternating sum of (co)face maps is denoted $s(U)$ and called the *Moore complex* of U.

(3.1.24) Proposition-definition. Let $\mathcal{F}$ be a presheaf of abelian groups on a site $\mathcal{C}$. Then $\mathcal{F}$ extends to a functor

$$\mathrm{SR}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Ab}, \qquad \{X_i\}_{i \in I} \mapsto \prod_{i \in I} \mathcal{F}(X_i)$$

Let X be an object of $\mathcal{C}$ and K a hypercover of X. The simplicial object K of $\mathrm{SR}(\mathcal{C})$ (remember that we have the forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$) gives rise, via this functor, to a cosimplicial object of Ab, which we denote $\mathcal{F}(K)$. The cohomology $H^i(s(\mathcal{F}(K)))$ of the Moore complex is called the *cohomology of X with respect to the hypercover K*.

Proof. To justify this definition, we only need to see that for a morphism $\{X_i\}_{i \in I} \to \{Y_j\}_{j \in J}$ in $\mathrm{SR}(\mathcal{C})$ given by $\alpha : I \to J$ and $\varphi_i : X_i \to Y_{\alpha(i)}$ for all $i \in I$, we have a well defined functorial morphism

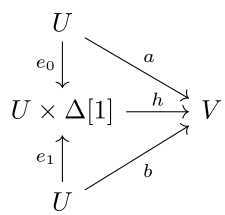
$$\prod_{j \in J} \mathcal{F}(Y_j) \to \prod_{i \in I} \mathcal{F}(X_i).$$

Defining such a map is equivalent to separately defining a map from the left hand side into $\mathcal{F}(X_i)$ for each i. So define that map as $(y_j)_{j \in J} \mapsto \mathcal{F}(\varphi_i)(y_{\alpha(i)})$. Functoriality is clear. □

3.1.4 Homotopies

(3.1.25) Definition: simplicial homotopies. Let U, V be simplicial objects of a category $\mathcal{C}$ and $a, b : U \to V$ two morphisms. Suppose the product $U \times \Delta[1]$ exists in $\mathcal{C}$.

- A *simplicial homotopy* from a to b is a morphism $h : U \times \Delta[1] \to V$ such that the diagram



  commutes, where e_0, e_1 are induced from the two maps $\Delta[0] \to \Delta[1]$ (recall that $U \times \Delta[0] = U$).

- We say that the morphisms a and b *are homotopic* or that a *is homotopic to* b if there is a finite sequence of morphisms $a = a_0, \dots, a_n = b$ such that for all $i = 1, \dots, n$, there is a homotopy from a_{i-1} to a_i or a_i to a_{i-1}.

- We say that $f : U \to V$ is a *homotopy equivalence* if there exists a

$$g : V \to U$$

  such that $f \circ g$ is homotopic to id_V and $g \circ f$ is homotopic to id_U. In that case we say that U and V are *homotopy equivalent.*

(3.1.26) An explicit description of homotopies. Suppose we have a homotopy h from a to b where $a, b : U \to V$ are maps of simplicial objects as above. Now, write

$$(\Delta[1])_n = \mathrm{Hom}_\Delta([n], [1]) = \{\alpha_0^n, \dots, \alpha_{n+1}^n\}$$

where α_i^n denotes the map $[n] \to [1]$ given by

$$j \mapsto \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j \geq i \end{cases}.$$

The homotopy $h : U \times \Delta[1] \to V$, being a morphism of simplicial objects, has components

$$h_n : (U \times \Delta[1])_n = \coprod_i U_n \cdot \alpha_i^n \to V_n$$

(where we write $U_n \cdot \alpha_i^n$ just to distinguish the $n+2$ copies of U_n). Let

$$h_{n,i} : U_n \to V_n$$

denote the restriction of the map h_n to the component of the coproduct corresponding to α_i^n, $i = 0, \ldots, n+1$. Then one can easily check that we have the following

(1) $h_{n,0} = b_n$ and $h_{n,n+1} = a_n$

(2) $d_j^n \circ h_{n,i} = h_{n-1,i-1} \circ d_j^n$ for $i > j$

(3) $d_j^n \circ h_{n,i} = h_{n-1} \circ d_n^j$ for $i \leq j$

(4) $s_j^n \circ h_{n,i} = h_{n+1,i+1} \circ s_j^n$ for $i > j$

(5) $s_j^n \circ h_{n,i} = h_{n+1,i} \circ s_j^n$ for $i \leq j$.

Conversely, given a collection of such maps $h_{n,i}$ satisfying (1)-(5) above, they define a morphism h which is a homotopy from a to b. This can be proven using the fact that giving a morphism of simplicial sets is equivalent to giving a map on each level, commuting with all faces and degeneracies.

(3.1.27) Simplicial homotopies and chain homotopies. Let $\mathcal{A}$ be an abelian category and let U and V be two simplicial objects of $\mathcal{A}$, with

$$a, b : U \to V$$

simplicial morphisms. Recall the functor s mapping a simplicial object to its Moore complex (see (3.1.23)). If h is a simplicial homotopy from a to b, then we want to deduce a chain homotopy $s(h)$ from $s(a)$ to $s(b)$ (note that $s(a)$ and $s(b)$ are simply a and b). We define $s(h)_n : U_n \to V_{n+1}$ by the formula

$$s(h)_n = \sum_{i=0}^{n} (-1)^{i+1} h_{n+1,i+1} \circ s_i^n.$$

For a calculation showing that this is indeed a chain homotopy from a to b, see [Sta21, Section 019Q].

Using this, we can show that homotopy equivalent simplicial objects map to chain homotopy equivalent complexes. We state this fact as a theorem:

(3.1.28) Theorem. Let $\mathcal{A}$ be an abelian category. Let U, V be simplicial objects of $\mathcal{A}$. If

$$a : U \to V$$

is a simplicial homotopy equivalence, then

$$s(a) : s(U) \to s(V)$$

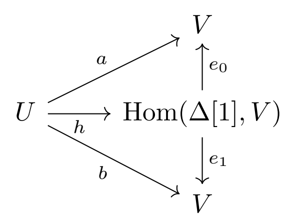
is a chain homotopy equivalence. □

(3.1.29) Definition - cosimplicial homotopies. Let U, V be cosimplicial objects of a category $\mathcal{C}$ and $a, b : U \to V$ two morphisms. Assume that the cosimplicial hom, $\mathrm{Hom}(\Delta[1], V)$, exists in $\mathcal{C}$ (a sufficient condition is that $\mathcal{C}$ have finite products).

- A (cosimplicial) *homotopy* from a to b is a morphism

  $$h : U \to \mathrm{Hom}(\Delta[1], V)$$

  such that the diagram

  commutes, where e_0, e_1 are induced from the two maps $\Delta[0] \to \Delta[1]$ (it is clear that $\mathrm{Hom}(\Delta[0], V) = V$.

- The concepts *homotopic* and *homotopy equivalence* are defined in exactly the same way as in the simplicial case.

(3.1.30) The duality of cosimplicial and simplicial homotopy. Let $\mathcal{C}$ be a category with finite products and let U, V be two cosimplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be a pair of morphisms of cosimplicial objects. The cosimplicial objects U and V correspond to simplicial objects U' and V' of $\mathcal{C}^{\mathrm{op}}$ and the morphisms a, b correspond to simplicial morphisms $a', b' : V' \to U'$. The existence of a cosimplicial homotopy h from a to b is equivalent to the existence of a simplicial homotopy from a' to b'. Indeed, this follows from the observation that finite products of $\mathcal{C}$ correspond to finite coproducts of $\mathcal{C}^{\mathrm{op}}$ and that $(\mathrm{Hom}(\Delta[1], V))' = V' \times \Delta[1]$, so the commutative diagram in the definition of a simplicial homotopy is the "primed" version of the one in the definition of a cosimplicial homotopy.

This observation allows us to deduce the theorem that we need later on:

(3.1.31) Theorem. Let $\mathcal{A}$ be an abelian category and U, V cosimplicial objects of $\mathcal{A}$. If $a : U \to V$ is a cosimplicial homotopy equivalence, then

$$s(a) : s(U) \to s(V)$$

is a homotopy equivalence of cochain complexes. □

(3.1.32) The boundary of $\Delta[n]$. Recall that $\Delta[n]$ has no nondegenerate m-simplices for $m > n$ and exactly one for $m = n$. The nondegenerate n-simplex is given by the identity,

$$\mathrm{id}_{[n]} \in (\Delta[n])_n = \mathrm{Hom}_{\Delta}([n],[n]).$$

For a simplicial set V, we have that $i_{m!}\,\mathrm{sk}_m\,V$ is the sub-simplicial set of V consisting of all i-simplices of V for $i \leq m$ and their degeneracies (see [Sta21, Lemma 018P] and [Sta21, Remark 018Q]). By analogy with topological n-simplices, we thus denote

$$\partial\Delta[n+1] = i_{n!}\,\mathrm{sk}_n\,\Delta[n+1]$$

and call it the *boundary*.

(3.1.33) Definition - trivial Kan fibrations. Let $X \to Y$ be a morphism of simplicial sets. It is called a *trivial Kan fibration* if $X_0 \to Y_0$ is surjective and for all $n \geq 1$ and for all commutative squares

$$\begin{array}{ccc} \partial\Delta[n] & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

a dotted arrow exists making the whole diagram commutative.

(3.1.34) Lemma. Let $X \to Y$ be a trivial Kan fibration of simplicial sets. Let $Z \to W$ be a termwise injective morphism of simplicial sets fitting into a commutative square as below. Then a dotted arrow exists making the whole diagram commutative.

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow \\ W & \longrightarrow & Y \end{array}$$

Proof. See [Sta21, Lemma 08NM]. □

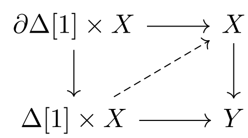
(3.1.35) Theorem. A trivial Kan fibration $f : X \to Y$ is a homotopy equivalence.

Proof. Applying (3.1.34) to the diagram

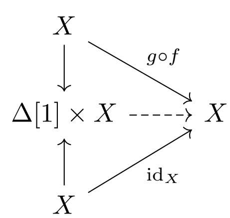
$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}_Y]{} & Y \end{array}$$

we see that f has a right inverse g. We then only need to show that $g \circ f$ is homotopic to id_X.

Now, $\partial\Delta[1]$ has its only nondegenerate simplices in degree 0, and there are two of them ($0 \mapsto 0$ and $0 \mapsto 1$). The degeneracies $Z_0 \to Z_n$ compose in a unique way for any simplicial set Z so we have that $(\partial\Delta[1])_n$ is a two-element set for all n. Thus $(\partial\Delta[1] \times X)_n = X_n \sqcup X_n$ for all n. We have a commutative square



where the top arrow is given by $\mathrm{id}_X = f \circ g$ and $g \circ f$ (using the characterisation of $\partial\Delta[1]$ above), the bottom arrow is given by f and the unique $\Delta[1] \to \Delta[0]$, the left arrow is given by the inclusion and id_X and the right one is f. The square is indeed commutative because $f \circ g \circ f = f$. By lemma (3.1.34) we can fill it in with the dotted arrow, which gives a homotopy from $g \circ f$ to id. More precisely, the top left triangle in our diagram above corresponds to the two triangles in the homotopy diagram



□

3.1.5 Hypercovers in the condensed setting

(3.1.36) Hypercovers in the site of compact Hausdorff spaces. In the case of condensed abelian groups, i.e. sheaves on the site CHaus, we want to show that for the purposes of cohomology, it is enough to consider only hypercovers that are singleton families on each level. So let S be a compact Hausdorff space and let K be a general hypercover of S. Write

$$K_j = \{S_{i,j} \to S\}_{i \in I_j}$$

and for each n,

$$(\mathrm{cosk}_n \, \mathrm{sk}_n \, K)_{n+1} = \{\tilde{S}_{j,n+1} \to S\}_{j \in J_{n+1}}.$$

We have that I_0 is finite and

$$\coprod_{i \in I_0} S_{i,0} \to S$$

is surjective. Further, each component of the canonical map

$$K_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n K)_{n+1}$$

gives a finite jointly surjective family. In other words, let $\alpha_n : I_{n+1} \to J_{n+1}$ be the map giving the canonical map above. Then $\alpha_n^{-1}(j)$ is a finite set for each $j \in J_{n+1}$. We want to show that I_{n+1} is finite by induction. We know that $(\operatorname{cosk}_n \operatorname{sk}_n K)_{n+1}$ is a finite limit in the category $\mathrm{SR}(\mathrm{CHaus}, S)$ of objects, whose indexing sets are finite by the inductive hypothesis. This means that the indexing set J_{n+1} of $(\operatorname{cosk}_n \operatorname{sk}_n K)_{n+1}$ is a subset of a finite product of finite sets and thus finite. This implies that I_{n+1} is finite.

We conclude that our hypercover K is of the form $K_n = \{S_{i,n} \to S\}_{i \in I_n}$ with I_n finite for all n. Further, $(\operatorname{cosk}_n \operatorname{sk}_n K)_{n+1} = \{\tilde{S}_{j,n+1} \to S\}_{j \in J_{n+1}}$ with J_{n+1} finite for all n, and we have a natural surjection

$$\coprod_{i \in I_{n+1}} S_{i,n+1} \to \coprod_{j \in J_{n+1}} \tilde{S}_{j,n+1}$$

We define a simplicial object $S_\bullet$ of CHaus by setting

$$S_n = \coprod_{i \in I_n} S_{i,n}$$

(since the families are finite, these disjoint unions remain compact Hausdorff). To show that $S_\bullet$ is in fact a hypercover of S, it suffices to show that

$$(\operatorname{cosk}_n \operatorname{sk}_n S_\bullet)_{n+1} = \coprod_{j \in J_{n+1}} \tilde{S}_{j,n+1}.$$

By the description of finite limits in (3.1.21), this follows from the general result that

$$\left(\coprod_{i=1}^{n} A_i\right) \times_C \left(\coprod_{j=1}^{m} B_j\right) = \coprod_{i,j} A_i \times_C B_j$$

in the category of sets (and thus also compact Hausdorff spaces).

Now if $\mathcal{F}$ from definition (3.1.24) is a condensed abelian group, then it takes finite coproducts to the corresponding finite products, and thus we see that the cohomology with respect to the two hypercovers is the same.

(3.1.37) Hypercovers of a point. Now consider a hypercover $S_\bullet$ of a point $*$ in the site CHaus, i.e. a simplicial object $S_\bullet$ of CHaus such that

$$S_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n S_\bullet)_{n+1}$$

is a surjection. We can regard $*$ as a constant simplicial set and thus the hypercover as a morphism of simplicial sets $S_\bullet \to *$. We want to show that this morphism is a trivial Kan fibration. By Yoneda for the left hand side and (3.1.19) for the right hand side, the above surjection translates into

$$\mathrm{Hom}_{\mathrm{Simp(Set)}}(\Delta[n+1], S_\bullet) \to \mathrm{Hom}_{\mathrm{Simp(Set)}}(\partial\Delta[n+1], S_\bullet)$$

being surjective, i.e. that any commutative square

$$\begin{array}{ccc} \partial\Delta[n+1] & \longrightarrow & S_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta[n+1] & \longrightarrow & * \end{array}$$

admits a dotted arrow making the whole diagram commute (the lower triangle being automatically commutative), and thus we conclude that $S_\bullet \to *$ is a trivial Kan fibration.

(3.1.38) Hypercovers of a finite set. Consider a hypercover $S_\bullet$ of a finite set S in the site CHaus. Then the fibres give a hypercover of each point separately. It is easy to see from the definition that a finite coproduct of homotopy equivalences is sitll a homotopy equivalence, and since the hypercover of each point is a trivial Kan Fibration by (3.1.37) and thus a homotopy equivalence by (3.1.35), the morphism of simplicial sets $S_\bullet \to S$ is a homotopy equivalence.

3.2 Cohomology

3.2.1 Different notions of cohomology

(3.2.1) Internal cohomology of the topos of condensed sets. A natural notion of cohomology of an object S of a site $\mathcal{C}$ is to take the higher right derived functors of the global sections functor $\Gamma(S,-)$ from the category of sheaves of abelian groups on $\mathcal{C}$ to the category of abelian groups. These are the cohomology groups of the right derived functor of $\Gamma(S,-)$, i.e.

$$R\Gamma(S, M).$$

Now suppose S is a compact Hausdorff space and regard condensed abelian groups as sheaves of abelian groups on the site CHaus. We have seen that

$$\Gamma(S,-) = \mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[\underline{S}], -)$$

so the cohomology with coefficients in the condensed abelian group M is given by the cohomology of the object

$$R\,\mathrm{Hom}(\mathbb{Z}[\underline{S}], M)$$

(here we have used the relationship between derived functors and derived bifunctors described in (2.5.14)). This is the sort of cohomology we are interested in relating to classical notions of cohomology in the present chapter. In the next chapter, however, we will use a slight natural generalisation. The above notion of cohomology is really a cohomology of the condensed set $\underline{S}$ associated to the topological space S. We might as well define the *cohomology of a condensed set T with coefficients in the condensed abelian group M* as the cohomology of

$$R\operatorname{Hom}(\mathbb{Z}[T], M).$$

These cohomology groups will be denoted

$$H^i_{\text{cond}}(T, M).$$

For a condensed abelian group A we denote the i-th cohomology group of $R\operatorname{Hom}(A, M)$ by $\operatorname{Ext}^i(A, M)$ so we have

$$H^i_{\text{cond}}(T, M) = \operatorname{Ext}^i(\mathbb{Z}[T], M)$$

(3.2.2) Remark. We can also define *internal Ext* as the cohomology objects

$$\underline{\operatorname{Ext}}^i(A, M) = H^i(R\underline{\operatorname{Hom}}(A, M)).$$

These of course belong to the category of condensed abelian groups. This notion won't be used until in chapter 4.

(3.2.3) A closer look at the condensed cohomology of S. For a compact Hausdorff space S, we can describe explicitly how to calculate the cohomology groups $H^i_{\text{cond}}(\underline{S}, M)$. We need to find a complex representing $R\operatorname{Hom}(\mathbb{Z}[\underline{S}], M)$, and this is done by either taking a projective resolution of $\mathbb{Z}[\underline{S}]$ or an injective resolution of M. We use the former approach and construct a projective resolution $P^\bullet \to \mathbb{Z}[\underline{S}]$ by using a simplicial hypercover $S_\bullet \to S$ by extremally disconnected sets and setting $P^i = \mathbb{Z}[\underline{S_i}]$.

To do this, let $S_0 \to S$ be a surjection where S_0 is an extremally disconnected set. Suppose we have defined $S_0, \ldots, S_n$ in such a way that they form an n-truncated simplicial object of ED. Then let S_{n+1} be the an extremally disconnected set surjecting onto $(\operatorname{cosk}_n(S_0, \ldots, S_n))_{n+1}$. Then we want to show that $S_0, \ldots, S_{n+1}$ form an $(n+1)$-truncated simplicial object of ED. Indeed, the face maps from S_{n+1} are obtained by composing the face maps from $\operatorname{cosk}_n(S_0, \ldots, S_n)$ with the surjection; the degeneracy maps into S_{n+1} are obtained by composing degeneracy maps into $\operatorname{cosk}_n(S_0, \ldots, S_n)$ with its section. It is easy to verify that these maps satisfy the required properties from proposition (3.1.4). Also, by definition the natural map $S_{n+1} \to \operatorname{cosk}_n(\operatorname{sk}_n(S_\bullet))_{n+1}$ is surjective. Being a simplicial object of ED, $S_\bullet$ is also a simplicial object of CHaus; thus $S_\bullet \to S$ is a hypercover with respect to the site CHaus with finite jointly surjective families as covers.

By [Sta21, Lemma 01GF], the Moore complex $\mathbb{Z}[S_\bullet]$ (with alternating sums of face maps) is exact. Further, we have seen that the free condensed abelian groups on extremally disconnected sets are projective, so we do indeed have a projective resolution of $\mathbb{Z}[\underline{S}]$.

The above shows that the cohomology groups $H^i_{\text{cond}}(\underline{S}, M)$ are then computed by the complex

$$0 \to \Gamma(S_0, M) \to \Gamma(S_1, M) \to \Gamma(S_2, M) \to \cdots$$

(3.2.4) Classical sheaf and Čech cohomology. A more classical notion of cohomology on the compact Hausdorff space S is the following: Let $\mathcal{F}$ be an abelian sheaf on the topological space S. The *i-th sheaf cohomology group* of S with respect to $\mathcal{F}$ is the i-th right derived functor of $\Gamma(S, -)$ at $\mathcal{F}$. Explicitly it is computed by taking an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ and computing the cohomology of the complex

$$0 \to \Gamma(S, \mathcal{I}^0) \to \Gamma(S, \mathcal{I}^1) \to \cdots$$

We denote these cohomology groups $H^i_{\text{sheaf}}(S, \mathcal{F})$. These cohomology groups can, when S is compact Hausdorff, be calculated as the Čech cohomology groups as well (see the discussion in the beginning of Lecture III of [Sch19b]).

3.2.2 Relating classical and condensed cohomology

Scholze attributes theorem (3.2.5) to Dyckhoff [Dyc76]. We give here the proof found in [Sch19b].

(3.2.5) Theorem. Let S be a compact Hausdorff space and M a discrete abelian group. There are natural isomorphisms

$$H^i_{\text{sheaf}}(S, M) \cong H^i_{\text{cond}}(S, \underline{M})$$

where on the left, M is regarded as the sheafification of the constant presheaf $U \mapsto M$.

Proof. We begin by treating the case where S is a profinite set. In this case, we want to show that for $i = 0$, both are equal to the set of continuous maps from S into M and for $i > 0$ both are 0.

The result for $H^i_{\text{sheaf}}(S, M)$. First, suppose $i = 0$. We have that

$$H^0_{\text{sheaf}}(S, M) = \Gamma_{\text{sheaf}}(S, M).$$

The sheafification of the constant presheaf M takes every open U to the set of locally constant maps $U \to M$, but since M is discrete, this is the same as continuous maps $U \to M$; in particular, $H^0_{\text{sheaf}}(S, M) = C(S, M)$.

For the case $i > 0$ write $S = \varprojlim_j S_j$ as a cofiltered limit of finite sets. We have $H^i_{\text{sheaf}}(S_j, M) = 0$ for all j, thus by [ES52, Chapter X, Theorem 3.1] and [God58, Théorème 5.10.1], we have $H^i_{\text{sheaf}}(S, M) = 0$.

For $H^i_{\text{cond}}(S, \underline{M})$, we take a simplicial hypercover $S_\bullet$ of S by extremally disconnected sets as described above. Then

$$H^0_{\text{cond}}(S, M) = \Gamma(S, \underline{M}) = \underline{M}(S) = C(S, M)$$

as desired.

For the case $i > 0$, we observe that by [Sta21, Lemma 03F9] it suffices to show that for any surjection $S' \to S$ of profinite sets, the Čech complex

$$0 \to \Gamma(S, M) \to \Gamma(S', M) \to \Gamma(S' \times_S S', M) \to \cdots$$

is exact. This follows from the case for S', S finite by passing to cofiltered limits (as we have seen, the global sections functor commutes with all limits and colimits).

Now for the general case; S a compact Hausdorff space. We denote by α_* the functor taking a sheaf $\mathcal{F}$ on the site of compact Hausdorff spaces to the sheaf $\alpha_*(\mathcal{F})$ on the topological space S defined such that for all $U \subset S$ open,

$$\alpha_*(\mathcal{F})(U) := \varprojlim_{U \supset V \text{ closed in } S} \mathcal{F}(V)$$

(this α_* arises from a natural morphism of topoi $\alpha : \mathrm{Sh}(\mathrm{CHaus}/S) \to \mathrm{Sh}(S)$). One easily sees that $\Gamma_{\text{cond}}(S, -) = \Gamma_{\text{sheaf}}(S, -) \circ \alpha_*$ and that α_*, being a limit, commutes with finite limits, and therefore is left exact. Thus by [Sta21, Lemma 015M], $R\Gamma_{\text{cond}}(S, -) = R\Gamma_{\text{sheaf}}(S, -) \circ R\alpha_*$. We want to show that $R\alpha_* M = M$ in the derived category of abelian sheaves on S, because then,

$$\begin{aligned} H^i_{\text{cond}}(S, M) &= H^i(R\Gamma_{\text{cond}}(S, M)) = H^i(R\Gamma_{\text{sheaf}}(S, R\alpha_* M)) \\ &= H^i(R\Gamma_{\text{sheaf}}(S, M)) = H^i_{\text{sheaf}}(S, M) \end{aligned}$$

as desired.

It suffices to show that for all $s \in S$, $(R\alpha_* M)_s = M$.

Since the open neighborhoods U of s are cofinal with the closed neighborhoods V of s (i.e. any open contains a closed neighborhood (take a compact one, it is closed)), we have

$$(R\alpha_* M)_s = \varinjlim_{U \ni s} R\Gamma_{\text{sheaf}}(U, R\alpha_* M) = \varinjlim_{U \ni s} R\Gamma_{\text{cond}}(U, M) = \varinjlim_{V \ni s} R\Gamma_{\text{cond}}(V, M).$$

Since the H^i_{cond} vanish for profinite sets whenever $i > 0$, then by the usual arguments for derived (bi)functors we can take a simplicial hypercover $S_\bullet \to S$ of *profinite* sets and compute $R\Gamma_{\text{cond}}(S, M)$ by the complex

$$0 \to \Gamma(S_0, M) \to \Gamma(S_1, M) \to \cdots$$

Note that however the face maps $S_i \to S$ are composed, we get the same result (follows by induction using the property $d_i d_j = d_{j-1} d_i$ and the base case $\operatorname{cosk}_0(S_0)_1 = S_0 \times_S S_0$). Thus the fibre product $S_i \times_S V$ is well defined and clearly $S_\bullet \times_S V$ is a simplicial hypercover of V. Therefore, the complex

$$0 \to \Gamma(S_0 \times_S V, M) \to \Gamma(S_1 \times_S V, M) \to \cdots$$

computes $R\Gamma_{\text{cond}}(V, M)$. Taking the filtered colimit of the complex above yields

$$0 \to \Gamma(S_0 \times_S \{s\}, M) \to \Gamma(S_1 \times_S \{s\}, M) \to \cdots$$

which computes $R\Gamma(\{s\}, M) = M$. Thus,

$$(R\alpha_* M)_s = \varinjlim_{V \ni s} R\Gamma_{\text{cond}}(V, M) = M$$

as desired. □

(3.2.6) Theorem. For any compact Hausdorff space S, let $C(S, \mathbb{R})$ denote the space of continuous real-valued functions as usual. Then

$$H^i_{\text{cond}}(S, \underline{\mathbb{R}}) = \begin{cases} C(S, \mathbb{R}) & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}.$$

Proof. This theorem is a corollary of theorem (3.2.7) below. □

(3.2.7) Theorem. Let S be a compact Hausdorff space. For any simplicial hypercover $S_\bullet \to S$ by profinite sets S_i, the complex of Banach spaces

$$(*) \qquad 0 \to C(S, \mathbb{R}) \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \cdots$$

satisfies the following "quantitative" version of exactness: if $f \in C(S_i, \mathbb{R})$ satisfies $df = 0$, then for any $\varepsilon > 0$ there exists a $g \in C(S_{i-1}, \mathbb{R})$ such that $dg = f$ and $\|g\| \leq (i + 2 + \varepsilon) \|f\|$ ($\|\cdot\|$ denotes the supremum norm).

Proof sketch. The theorem is proved by a series of lemmas:

(3.2.8) Lemma. If S and all S_i are finite, then $(*)$ is exact and if $f \in C(S_i, \mathbb{R})$ satisfies $df = 0$, then there is a $g \in C(S_{i-1}, \mathbb{R})$ with $\|g\| \leq (i+2) \|f\|$ and $dg = f$.

Proof. In this case, the morphism $S_\bullet \to S$ of simplicial sets is a homotopy equivalence by (3.1.38). Denote it by α and its homotopy inverse by β. Also, since the S_i and S are all finite, all maps from them into $\mathbb{R}$ are continuous. Regarding $\mathbb{R}$ as a constant cosimplicial object, we see that the cosimplicial abelian group given by the $C(S_i,\mathbb{R})$'s is just the cosimplicial abelian group $\mathrm{Hom}(S_i,\mathbb{R})$. Since

$$\begin{aligned}\mathrm{Hom}(\Delta[1]\times S_\bullet,\mathbb{R})_n &= \mathrm{Hom}_{\mathrm{Set}}(\Delta[1]_n\times S_n,\mathbb{R})\\ &= \mathrm{Hom}_{\mathrm{Set}}(\Delta[1]_n,\mathrm{Hom}_{\mathrm{Set}}(S_n,\mathbb{R}))\\ &= \mathrm{Hom}(\Delta[1],\mathrm{Hom}(S_\bullet,\mathbb{R})_n\end{aligned}$$

we have

$$\mathrm{Hom}(\Delta[1]\times S_\bullet,\mathbb{R}) = \mathrm{Hom}(\Delta[1],\mathrm{Hom}(S_\bullet,\mathbb{R}))$$

and thus the simplicial homotopy diagram is transformed to a cosimplicial homotopy diagram and we have a cosimplicial homotopy equivalence from $C(S_\bullet,\mathbb{R})$ to the constant simplicial abelian group $C(S,\mathbb{R})$. This gives a chain homotopy equivalence between the Moore complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(S_0,\mathbb{R}) & \longrightarrow & C(S_1,\mathbb{R}) & \longrightarrow & C(S_2,\mathbb{R}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(S,\mathbb{R}) & \xrightarrow{\mathrm{id}} & C(S,\mathbb{R}) & \xrightarrow{0} & C(S,\mathbb{R}) & \xrightarrow{\mathrm{id}} & \cdots \end{array}$$

In the following, for a map $\varphi : A\to B$, φ^* will denote the map

$$C(B,\mathbb{R})\to C(A,\mathbb{R}),\quad f\mapsto f\circ\varphi.$$

Now take $f\in C(S_i,\mathbb{R})$ with $df=0$. We have a homotopy given by

$$h_n : C(S_n,\mathbb{R})\to C(S_{n-1},\mathbb{R})$$

satisfying

$$d\circ h_n + h_{n+1}\circ d = \mathrm{id}_{C(S_n,\mathbb{R})} - \alpha_n^*\circ\beta_n^*$$

for all n. Therefore,

$$d(h_i(f)) = f - \alpha_i^*(\beta_i^*(f)).$$

If i is even, then since β^* is a morphism of comlpexes, we have $\mathrm{id}\circ\beta_i^* = \beta_{i+1}^*\circ d$ and thus $\beta_i^*(f) = \beta_{i+1}^*(d(f)) = 0$. We conclude in the case i even that

$$f = d(h_i(f)).$$

In the case i odd, we use that α^* is a morphism of complexes, i.e.

$$\alpha_i^*\circ\mathrm{id} = d\circ\alpha_{i-1}^*.$$

We condlude in the case where i is odd that

$$f = d\left(h_i(f) + \alpha_{i-1}^*(\beta_i^*(f))\right).$$

Since $\alpha_{i-1}^*(\beta_i^*(f)) = f\circ\beta_i\circ\alpha_i$, we have $\left\|\alpha_{i-1}^*(\beta_i^*(f))\right\| \le \|f\|$. By (3.1.27) and the correspondence between simplicial and cosimplicial homotopies, $h_i(f)$ is an alternating sum of $i+1$ pullbacks of f. Therefore $\|h_i(f)\|\le (i+1)\|f\|$. In either case i odd or even, we conclude that $f = dg$ with $\|g\|\le(i+2)\|f\|$. □

(3.2.9) Remark. Let N be a normed vector space. Then the following is well known: There exists a Banach space B, unique up to isometry, such that N can be isometrically and densely embedded in B. We call B the *completion* of N and regard N as a subet of B.

(3.2.10) Lemma. Let

$$0 \to N^0 \to N^1 \to N^2 \to N^3 \to \cdots$$

be a complex of normed vector spaces with continuous differential, satisfying the following quantitative version of exactness: For each i, there is a constant M_i such that for every $f \in N^i$ such that $df = 0$, for any ε there is a $g \in N^{i-1}$ with $dg = f$ and $\|g\| \leq (M_i + \varepsilon)\,\|f\|$. Then the corresponding complex of the completions satifies the same quantitative exactness.

Proof. Recall that a sequence $(f_n)_{n\in\mathbb{N}}$ is said to be *rapidly Cauchy* if the series

$$\sum_{n\in\mathbb{N}} \|f_{n+1} - f_n\|$$

is convergent. Let C^i be the space of rapidly Cauchy sequences of N^i and Z^i the subspace of C^i consisting of null sequences (sequences tending to zero). Then it can be shown that the completion of N^i is the quotient C^i/Z^i with the norm of $[(x_n)_{n\in\mathbb{N}}]$ given by $\lim_{n\to\infty} \|x_n\|$ (which is well defined since the sequence of norms is a Cauchy sequence in $\mathbb{R}$ and thus convergent).

To show that the complex of completions is exact, we prove that $C^\bullet$ and $Z^\bullet$ are exact. Then using that $0 \to Z^\bullet \to C^\bullet \to C^\bullet/N^\bullet \to 0$ is a short exact sequence of complexes, we pass to the long exact sequence in cohomology and deduce that $C^\bullet/N^\bullet$ is exact.

For exactness of $Z^\bullet$: Let (f_n) be a null sequence in Z^i with differential zero. Let $\varepsilon > 0$. Then for every n there exists g_n with $dg_n = f_n$ and

$$\|g_n\| \leq (M_i + \varepsilon)\,\|f_n\| \to 0 \text{ when } n \to \infty.$$

Thus (g_n) is a null sequence and we conclude that $Z^\bullet$ is exact.

For exactness of $C^\bullet$: Let (f_n) be a rapidly Cauchy sequence in C^i with differential zero. Let $\varepsilon > 0$. For convenience, let $f_{-1} = 0$. Since $d(f_n - f_{n-1}) = 0$, there exists g_n with $dg_n = f_n - f_{n-1}$ and $\|g_n\| \leq (M_i + \varepsilon)\,\|f_n - f_{n-1}\|$ for all $n \geq 0$. Let

$$h_m = \sum_{i=0}^{m} g_i$$

Then $dh_m = f_m$, and (h_n) is rapidly Cauchy, since

$$\sum_{n\in\mathbb{N}} \|h_{n+1} - h_n\| = \sum_{n\in\mathbb{N}} \|g_{n+1}\| \leq (M_i + \varepsilon) \sum_{n\in\mathbb{N}} \|f_{n+1} - f_n\| < +\infty$$

We conclude that the complex of completions is exact. It remains to show that it satisfies the quantitative version of exactness. So suppose $df = 0$. Let (f_n) be a rapidly Cauchy sequence in $\operatorname{Ker} d$ converging to f. For each n take g_n with $dg_n = f_n$ and $\|g_n\| \leq (M_i + \varepsilon)\|f_n\|$. Then passing to the limits (since d is continuous), we obtain the desired result for g and f. □

(3.2.11) Lemma. Any profinite set $X = \varprojlim_i X_i$, where the X_i are finite sets, is isomorphic to a $\varprojlim_i X'_i$ with each X'_i finite, in which all the transition maps are surjective.

Proof. By (A.1.7) we can assume that X is the limit of a directed inverse system (X_i, f_{ij}) indexed by a partially ordered set I, where $f_{ij} : X_j \to X_i$ whenever $i \leq j$. We will define another directed inverse system (X'_i, f'_{ij}) with all the f'_{ij} surjective, which has the same limit X.

Define

$$X'_i = \bigcap_{j \geq i} f_{ij}(X_j)$$

It is clear that for all i, $X'_i \subset X_i$. For $j \geq i$, define $f'_{ij} : X'_j \to X'_i$ as the restriction of f_{ij}. It is not a priori clear that f'_{ij} lands in X'_i. To show this, let

$$x \in X'_j = \bigcap_{k \geq j} f_{jk}(X_k).$$

We want to show that $f'_{ij}(x) = f_{ij}(x) \in X'_i$, i.e. that for all $k \geq i$, there is a $y \in X_k$ such that $f_{ik}(y) = f_{ij}(x)$. Since we are working in a directed inverse system, pick an $s \in I$ with $s \geq j$ and $s \geq k$. Since $x \in X'_j$, there is a $z \in X_s$ with $f_{js}(z) = x$. Set $y = f_{ks}(z)$; then we have

$$f_{ik}(y) = f_{ik}(f_{ks}(z)) = f_{is}(z) = f_{ij}(f_{js}(z)) = f_{ij}(x)$$

as desired.

It is now clear that we have a directed inverse system (X'_i, f'_{ij}). Next up is showing that for $j \geq i$, $f'_{ij} : X'_j \to X'_i$ is surjective. Let $x \in X'_i$. Then for all $k \geq i$, there is an $x_k \in X_k$ with $f_{ik}(x_k) = x$, in particular the fibre $f_{ik}^{-1}\{x\}$ is nonempty. It suffices to show that there exists an element $y \in f_{ij}^{-1}\{x\}$ with $y \in X'_j$. For a contradiction, suppose there is no such y. Then for every $y \in f_{ij}^{-1}\{x\}$ there is an index $j_y \geq j$ such that $f_{jj_y}^{-1}\{y\} = \varnothing$. Now since $f_{ij}^{-1}\{x\} \subset X_i$ is finite, we can find a $k \in I$ such that $k \geq j_y$ for all y. Then we have

$$x = f_{ik}(x_k) = f_{ij}(f_{jk}(x_k))$$

so

$$f_{jk}(x_k) \in f_{ij}^{-1}\{x\}$$

which means that $f_{jk}(x_k)$ is one of the y's above, so write

$$y = f_{jk}(x_k) = f_{jj_y}(f_{j_y k}(x_k))$$

and conclude that

$$f_{j_y k}(x_k) \in f_{jj_y}^{-1}\{y\} = \varnothing$$

and we have our contradiction.

Finally, we show that the two inverse systems (X_i, f_{ij}) and (X_i', f_{ij}') have the same limit. The limit of the former is

$$\left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i : f_{ij}(x_j) = x_i \text{ for all } j \geq i \right\}$$

and from the definition we see that each x_i actually lies in X_i', and since f_{ij}' is just the restriction of f_{ij} we conclude that this limit actually equals

$$\left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i' : f_{ij}'(x_j) = x_i \text{ for all } j \geq i \right\}$$

which is precisely the limit of the inverse system (X_i', f_{ij}'). □

We state the next two (important) lemmas without proof.

(3.2.12) Lemma. If $X = \varprojlim_i X_i$ is a profinite set (with each X_i finite), then $C(X, \mathbb{R})$ is the completion of $\varinjlim_i C(X_i, \mathbb{R})$. □

(3.2.13) Lemma. If S_i and S in the theorem are all profinite, the statement of the theorem is true. □

We pass to the most general case, $S \in$ CHaus. Let $S_\bullet \to S$ be a hypercover by profinite sets S_i.

Observation: For any $s \in S$, the sets $S \times_S \{s\} = \{s\}$ and $S_i \times_S \{s\}$ are profinite so the corresponding complex

$$0 \to C(S \times_S \{s\}, \mathbb{R}) \to C(S_0 \times_S \{s\}, \mathbb{R}) \to C(S_1 \times_S \{s\}, \mathbb{R}) \to \cdots$$

satisfies the condition in the last lemma.

Now let $f \in C(S_i, \mathbb{R})$, $df = 0$. Let $s \in S$. We can regard $S_i \times_S \{s\}$ as the fibre of s along $S_i \to S$, and in particular talk about the restriction of f to $S_i \times_S \{s\}$, which we denote $f_s \in C(S_i \times_S \{s\}, \mathbb{R})$. By the observation above and lemma (3.2.13), there is a $g_s \in C(S_{i-1} \times_S \{s\}, \mathbb{R})$ with $dg_s = f_s$ and $\|g_s\| \leq (i+2+\varepsilon)\, \|f_s\|$. By Tietze's extension theorem, there is a $\tilde{g}_s \in C(S_{i-1}, \mathbb{R})$

extending g_s with $\|\tilde{g_s}\| \leq \|g_s\|$. Since S_i is compact Hausdorff, the closed subset $S_i \times_S \{s\}$ of S_i on which $d\tilde{g_s} - f$ is zero can be separated from the closed subset consisting of the elements where $d\tilde{g_s} - f$ takes values $\geq \varepsilon \|f\|$. In particular, we can find an open neighbourhood U_s of s in S such that

$$\|(d\tilde{g_s} - f)|_{S_i \times_S U_s}\| \leq \varepsilon \|f\|$$

There is a finite subcover of $(U_s)_{s \in S}$; in other words there are opens $U_1, \ldots, U_n$ covering S and $g_j \in C(S_{i-1}, \mathbb{R})$ for $j = 1, \ldots, n$ with $\|g_j\| \leq (i+2+\varepsilon)\|f\|$ such that

$$\left\|(dg_j - f)|_{S_i \times_S U_j}\right\| \leq \varepsilon \|f\|$$

for all $j = 1, \ldots, n$.

Take a partition of unity $1 = \sum_{j=1}^n \rho_j$ with $0 \leq \rho_j \leq 1$ and $\operatorname{supp} \rho_j \subset U_j$. Let

$$g^{(0)} = \sum_{j=1}^n \rho_j g_j.$$

Then

$$\left\|g^{(0)}\right\| \leq (i+2+\varepsilon)\|f\|.$$

Set $f^{(1)} = f - dg^{(0)}$. We have

$$\left\|f^{(1)}\right\| = \left\|dg^{(0)} - f\right\| \leq \varepsilon \|f\|.$$

Start this process over with $f^{(1)}$ instead of f, and repeat, and we conclude

$$f = d(g^{(0)} + g^{(1)} + \cdots) =: dg$$

with

$$\left\|g^{(m)}\right\| \leq (i+2+\varepsilon)\left\|f^{(m)}\right\| \leq (i+2+\varepsilon)\varepsilon^m \|f\|$$

so

$$\|g\| \leq \frac{i+2+\varepsilon}{1-\varepsilon}\|f\|$$

as desired (redefining ε). $\square$

Chapter 4

Locally compact abelian groups and their associated condensed abelian groups

We begin this chapter in section 4.1 by discussing further the relationship between topological spaces and condensed sets, as well as introducting topological groups and in particular locally compact abelian groups. section 4.2 gives a very short introduction to spectral sequences, stating the main theorem we need, when calculating derived homs between locally compact abelian groups in section 4.3

4.1 Some topology

4.1.1 Condensed vs. topological

It is time to discuss the relationship between topological spaces and condensed sets in more detail than in chapter 1.

(4.1.1) Compactly generated weak Hausdorff spaces. A subcategory of topological spaces often used by homotopy theorists is CGWH, which we define below. This category has some nicer properties, such as being *cartesian closed*, a property not satisfied by the whole category of topological spaces.

A category is said to be *cartesian closed* if it has products of pairs and an internal hom satisfying

$$\mathrm{Hom}(X \times Y, Z) = \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z)).$$

Topological spaces have internal hom: the set of continuous maps $X \to Y$ with the compact-open topology. Products of pairs of course exist as well (product topology), but the adjunction above is only satisfied in certain special cases (such as Y being locally compact).

A topological space X is said to be *compactly generated* if continuous maps $X \to Y$ are precisely those making the composite $S \to X \to Y$ continuous for every compact Hausdorff space S mapping continuously to X.

A topological space X is *weak Hausdorff* if for any compact Hausdorff S with a continuous $f : S \to X$, the image $f(S)$ is (compact) Hausdorff.

We will in the following mainly be interested of the former property.

(4.1.2) Remark. We say that a topological space X is κ*-compactly generated* if it satisfies the condition in the definition of compactly generated for every κ-small compact Hausdorff space S, instead of every compact Hausdorff space S. We want to prove the following: a κ-small topological space X is compactly generated if and only if it is κ-compactly generated.

Proof. We need to show that if $f : X \to Y$ has the property that for every κ-small compact Hausdorff space S' with a continuous $g' : S' \to X$, $f \circ g'$ is continuous, then for every compact Hausdorff space S with a continuous $g : S \to X$, $f \circ g$ is continuous. But we can just take S' to be the quotient $S/\sim$ where $s_1 \sim s_2$ if and only if $g(s_1) = g(s_2)$. Then $|S| \leq |X| < \kappa$ and since the equivalence relation $\sim$ is closed and S is compact Hausdorff, S' is compact Hausdorff. Then indeed, we have an induced continuous $g' : S' \to X$ and thus $f \circ g'$ is continuous. Let $q : S \to S'$ denote the quotient map; then $f \circ g = f \circ g' \circ q$ is continuous. □

(4.1.3) Lemma. The inclusion of the category of compactly generated spaces in topological spaces admits a right adjoint $X \mapsto X^{\mathrm{cg}}$, where X^{cg} has underlying set X and the topology is the final topology for the collection of all continuous maps $S \to X$ where S is a compact Hausdorff space (i.e. the finest topology on the set X making them all continuous).

Proof. Note that X^{cg} is compactly generated by definition. For the adjunction, we need to show that if Y is compactly generated, then $f : Y \to X$ is continuous if and only if $f : Y \to X^{\mathrm{cg}}$ is continuous. Since the topology on X^{cg} is finer than the one on X, the $\Leftarrow$ direction is obvious. Suppose $f : Y \to X$ is continuous. Since Y is compactly generated, to show that $f : Y \to X^{\mathrm{cg}}$ is continuous, it suffices to show that for any compact Hausdorff space S with a continuous $g : S \to Y$, the composite $f \circ g : S \to X^{\mathrm{cg}}$ is continuous. Since $f \circ g : S \to X$ is continuous, this is true by definition, and we are done. □

(4.1.4) Remark. Remark (4.1.2) shows that lemma (4.1.3) holds even if the category of topological spaces is replaced with the category of κ-small topological spaces.

(4.1.5) Theorem. (i) The functor $X \mapsto \underline{X}$ from (κ-small) topological spaces to condensed sets is faithful, and fully faithful when restricted to the subcategory of compactly generated topological spaces.

(ii) The functor $X \mapsto \underline{X}$ admits a left adjoint $T \to T(*)$ where the underlying set $T(*)$ is equipped with the final topology for the collection of all maps $S \to T(*)$ where S is compact Hausdorff, that come from a map of condensed sets $\underline{S} \to T$ (here we regard condensed sets as sheaves on the site CHaus).

Proof. We start by proving that (ii) implies (i). Note that $\underline{X}(*) = X^{\mathrm{cg}}$ (indeed, morphisms of condensed sets $\underline{S} \to \underline{X}$ are, by Yoneda, in bijection with $\underline{X}(S)$, which are precisely the continuous maps $S \to X$). Thus by the adjunction in (ii), we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CondSet}}(\underline{X}, \underline{Y}) &= \mathrm{Hom}_{\mathrm{Top}}(\underline{X}(*), Y) \\ &= \mathrm{Hom}_{\mathrm{Top}}(X^{\mathrm{cg}}, Y) \\ &\hookrightarrow \mathrm{Hom}_{\mathrm{Top}}(X, Y) \end{aligned}$$

where the arrow at the bottom is an isomorphism if X is compactly generated, i.e. $X = X^{\mathrm{cg}}$.

Now we prove (ii). Let T be a condensed set and X a topological space (recall that we are working in κ-condensed sets and κ-small topological spaces). We want to show the functorial isomorphism

$$\mathrm{Hom}_{\mathrm{CondSet}}(T, \underline{X}) = \mathrm{Hom}_{\mathrm{Top}}(T(*), X).$$

It suffices to show that giving a morphism of condensed sets $T \to \underline{X}$ is equivalent to giving a map of sets $T(*) \to X$ such that for each compact Hausdorff space S with a map of condensed sets $\underline{S} \to T$, the composite $S \to T(*) \to X$ is continuous. By Yoneda, the maps $\underline{S} \to T$ are in bijection with the set $T(S)$ and thus taking such a map to the induced composite $S \to T(*) \to X$ gives a map of condensed sets $T \to \underline{X}$. Suppose then that we have a map of condensed sets $f : T \to \underline{X}$. Let S be a compact Hausdorff space with a map of condensed sets $\eta : \underline{S} \to T$. Then $f \circ \eta : \underline{S} \to \underline{X}$ is a map of condensed sets and corresponds by Yoneda to an element of $\underline{X}(S)$, i.e. a continuous map $S \to X$, namely $f \circ \eta(\mathrm{id}_S)$. We need to show that this map is the same as the map $S \to T(*) \to X$. But this follows from considering each of the naturality diagrams

$$\begin{array}{ccc} \underline{S}(S) & \longrightarrow & \underline{X}(S) \\ \downarrow & & \downarrow \\ \underline{S}(*) & \longrightarrow & \underline{X}(*) \end{array}$$

corresponding to each of the constant maps $* \to S$, for the natural transformation $f \circ \eta$, which by definition factors through T, and we are done. □

4.1.2 Quasi-compact and quasi-separated condensed sets

Here, we give a result identifying compact Hausdorff spaces with a certain class of condensed sets. Although not necessary for what follows, the result (4.1.8) fits the spirit of this section.

(4.1.6) Definition. A sheaf $\mathcal{F}$ on a site $\mathcal{C}$ is said to be *quasi-compact* (abbreviated qc) if for any collection of maps of sheaves $\{\mathcal{F}_i \to \mathcal{F}\}_{i\in I}$ such that the induced map

$$\coprod_{i\in I} \mathcal{F}_i \to \mathcal{F}$$

is an epimorphism, there exists a finite subset $J \subset I$ such that

$$\coprod_{i\in J} \mathcal{F}_i \to \mathcal{F}$$

is an epimorphism.

(4.1.7) Definition. A sheaf $\mathcal{F}$ on a site $\mathcal{C}$ is said to be *quasi-separated* (abbreviated qs) if it satisfies the following condition. For any $\mathcal{G}, \mathcal{H} \to \mathcal{F}$ with $\mathcal{G}$ and $\mathcal{H}$ quasi-compact sheaves, the sheaf $\mathcal{G} \times_{\mathcal{F}} \mathcal{H}$ is again quasi-compact. We use the abbreviation qcqs for sheaves that are quasi-compact and quasi-separated.

(4.1.8) Theorem. ([Sch19b, Theorem 2.16]).

(i) The functor $X \mapsto \underline{X}$ gives an equivalence from the category of compact Hausdorff spaces to the category of qcqs condensed sets.

(ii) A compactly generated space X is weak Hausdorff if and only if the condensed set $\underline{X}$ is qs. For any qs condensed set T, the topological space $T(*)$ is compactly generated weak Hausdorff.

□

4.1.3 Locally compact abelian groups

(4.1.9) Definition. An abelian group A equipped with a topology for which the addition and inverse

$$\begin{aligned} A \times A \to A, &\qquad (a,b) \mapsto a+b \\ A \to A, &\qquad a \mapsto -a \end{aligned}$$

are continuous, is called a *topological abelian group*.

(4.1.10) Definition. A topological abelian group A is called a *locally compact abelian group* or *LCA* if its underlying topological space is Hausdorff and locally compact. We also denote by LCA the category of locally compact abelian groups and continuous maps.

(4.1.11) Structure theorem for locally compact abelian groups. Let A be a locally compact abelian group. There exists an integer n and a locally compact abelian group A' admitting a compact open subgroup such that

$$A \simeq \mathbb{R}^n \times A'.$$

□

(4.1.12) Remark. The group A' in the structure theorem (4.1.11) is an extension of a discrete abelian group by a compact abelian group. Indeed, let K be the compact open subgroup. Then we need to show that A'/K is discrete. The cosets $x + K \subset A'$ are open for all $x \in A'$. This means that the singletons $\{x + K\} \subset A'/K$ are open in the quotient topology. Since all elements of A'/K have this form, this is a discrete abelian group.

(4.1.13) The circle $\mathbb{T} = \{e^{2\pi i\theta} : \theta \in \mathbb{R}\}$ is a compact subgroup of the locally compact abelian group $\mathbb{C}$. There is a natural isomorphism between $\mathbb{T}$ and the quotient $\mathbb{R}/\mathbb{Z}$.

(4.1.14) Pontrjagin duality. Let $\mathbb{T}$ denote the circle group $\mathbb{R}/\mathbb{Z}$. The functor $\mathbb{D}$, which takes a locally compact abelian group to the abelian group $\operatorname{Hom}(A, \mathbb{T})$ equipped with the compact-open topology, takes values in LCA and induces a contravariant autoequivalence of LCA. The map $A \to \mathbb{D}(\mathbb{D}(A))$ is an isomorphism. Moreover, $\mathbb{D}$ restricts to a duality from compact abelian groups to discrete abelian groups. □

(4.1.15) Remark. For proofs of the above results on locally compact abelian groups, see for example [DE09, Chapter 4].

(4.1.16) Remark. The category of locally compact abelian groups is not abelian. However, it comes close, and is what is called *quasi-abelian*. The kernel of a map $f : A \to B$ in LCA is the usual algebraic kernel equipped with the subspace topology. The cokernel of f is $B/\overline{\operatorname{Im} f}$ equipped with the quotient topology (it is a quotient by a closed equivalence relation and thus locally compact Hausdorff, and so belongs to LCA). The problem is that we can have maps with trivial kernel and cokernel that are not isomorphisms, for example the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$. The property of being quasi-abelian is still enough to define a bounded derived category $D^b(\mathrm{LCA})$, but there are neither enough injectives nor enough projectives. As we have seen, condensed abelian groups form a very nice abelian category, and we can embed $D^b(\mathrm{LCA})$ in $D(\mathrm{CondAb})$. In $D^b(\mathrm{LCA})$, there is a notion of derived hom, and in the rest of this chapter we give a proof sketch of the result that these derived homs can be calculated by the $R\underline{\operatorname{Hom}}$ that we have defined in $D(\mathrm{CondAb})$.

(4.1.17) Proposition. A locally compact Hausdorff space is compactly generated. In particular, locally compact abelian groups are compactly generated.

Proof. Let X be a locally compact Hausdorff space, and let $f : X \to Y$ be a map where Y is a topological space, such that for any compact Hausdorff space S with a continuous map $S \to X$, the composition $S \to X \to Y$ is continuous. We need to show that f is continuous. It suffices to show that it is continuous on a neighbourhood of every $x \in X$. Take $S \hookrightarrow X$ a compact neighbourhood of x and we are done. □

4.2 Spectral sequences

(4.2.1) Definition. Let r_0 be an integer ≥ 1. A *spectral sequence* in an abelian category $\mathcal{A}$ is a collection

$$(E_r^{p,q}, d_r^{p,q}, E^n)_{p,q,r,n\in\mathbb{Z},r\geq r_0}$$

where all the $E_r^{p,q}$ and E^n are objects of $\mathcal{A}$ and the

$$d_r^{p,q} : E_r^{p,q} \to E^{p+r,q-r+1}$$

are morphisms in $\mathcal{A}$.

For a fixed r, the collection of objects $(E_r^{p,q})_{p,q\in\mathbb{Z}}$ is called the *r-th page* of the spectral sequence. The morphisms $d_r^{p,q}$ are called *differentials*.

These data are required to satisfy the following conditions:

(i) For all p, q, r we have
$$d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0.$$

(ii) There are given isomorphisms
$$E_{r+1}^{p,q} \simeq H^0(E^{p+\bullet r,q-\bullet r+\bullet}).$$

(iii) For any pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ there is an r_∞ such that for all $r \geq r_\infty$, we have $d^{p,q} = d^{p-r,q+r-1} = 0$ (i.e. both the differentials into and out of $E_r^{p,q}$ are zero). In particular, $E_r^{p,q} \simeq E_{r_\infty}^{p,q} =: E_\infty^{p,q}$.

(iv) There is a *decreasing filtration*
$$\cdots \hookrightarrow F^{p+1}E^n \hookrightarrow F^pE^n \hookrightarrow \cdots \hookrightarrow E^n$$
such that
$$\bigcap_{p\in\mathbb{Z}} F^pE^n = 0 \qquad \text{and} \qquad \bigcup_{p\in\mathbb{Z}} F^pE^n = E^n,$$ [1]

[1] The meaning of the intersection (resp. union) of such a filtration in an arbitrary category should be clear: the limit (resp. colimit) over the totally ordered set $\mathbb{Z}$.

and there are given isomorphisms

$$E_\infty^{p,q} \simeq F^p E^{p+q}/F^{p+1}E^{p+q}.$$

(4.2.2) Remark. Note that the r_0-th page of a spectral sequence determine the rest of the pages. A spectral sequence is usually given by writing

$$E_{r_0}^{p,q} \Rightarrow E^{p+q}.$$

(4.2.3) Cohomology of the total complex. Suppose we have a double complex $A^{\bullet\bullet}$ in an abelian category $\mathcal{A}$ with the property that for all n,

$$A^{n-k,k} = 0$$

for $|k|$ sufficiently large. Then the two notions of total complex coincide and we denote by $A^\bullet$ the total complex which we recall is given by

$$A^n = \bigoplus_{i+j=n} A^{i,j} = \bigoplus_{j\in\mathbb{Z}} A^{n-j,j}.$$

We want to define a spectral sequence to compute the cohomology of the total complex. Let's clarify some notation. Recall that the vertical differentials are denoted d_v and the horizontal ones are denoted d_h. We denote the i-th cohomology of the total complex by $H^i(A^\bullet)$. By taking the p-th cohomology of each row, ($H^p(A^{\bullet,j})$ being the p-th cohomology group of the j-th row), since d_v is a morphism of complexes $A^{\bullet,j} \to A^{\bullet,j+1}$ for each j, we get another complex $H_h^p(A^{\bullet\bullet})$ with differential $H^p(d_v)$. Then we can take the q-th cohomology of this vertical complex, and denote it

$$E_2^{p,q} = H_v^q(H_h^p(A^{\bullet\bullet})).$$

As the notation suggests, these objects will form the second page of our spectral sequence. It also has an E_1-page, as stated in (4.2.4). We refer to [Huy06] for the precise construction, in particular the filtration.

(4.2.4) Theorem. ([Huy06, Proposition 2.64 and Remark 2.65]). Let $A^{\bullet\bullet}$ be a double complex like in (4.2.3) (and all the other notation defined there). Then there is a spectral sequence

$$E_2^{p,q} = H_v^q(H_h^p(A^{\bullet\bullet})) \Rightarrow H^{p+q}(A^\bullet).$$

It also has an E_1-page:

$$E_1^{p,q} = H^q(A^{\bullet,p}) \Rightarrow H^{p+q}(A^\bullet).$$

□

(4.2.5) An application. We now have the tools to prove certain results used in chapter 2, namely that a bounded above complex of projective condensed abelian groups is K-flat and K-projective in $K^-(\text{CondAb})$ and that a bounded below complex of injective condensed abelian groups is K-injective in $K^+(\text{CondAb})$ (cf. (2.6.2) and (2.5.9)). The proofs are all similar and we only prove the first one. So let $P \in K^-(\text{CondAb})$ consist of projective condensed abelian groups. Projective condensed abelian groups are flat so if M is an acyclic bounded above complex of condensed abelian groups, then for each j, the complex $M \otimes P^j$ is acyclic. Thus the E_1 page of associated spectral sequence vanishes, and thus also the limiting terms, i.e. the cohomology of the total complex. We conclude that the tensor product complex $M \otimes P$ is acyclic. □

4.3 The condensed viewpoint

(4.3.1) Proposition. Let A and B be Hausdorff topological groups with A compactly generated. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\text{Hom}}(\underline{A}, \underline{B}) \simeq \underline{\text{Hom}(A, B)}$$

where $\text{Hom}(A, B)$ is equipped with the compact-open topology.

Proof. We need to construct a map of condensed abelian groups

$$\eta : \underline{\text{Hom}}(\underline{A}, \underline{B}) \to \underline{\text{Hom}(A, B)}$$

We have

$$\underline{\text{Hom}}(\underline{A}, \underline{B})(S) = \text{Hom}_{\text{CondAb}}(\underline{A} \otimes \mathbb{Z}[\underline{S}], \underline{B}).$$

A map

$$f : \underline{A} \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$$

gives a map

$$f : A \otimes \mathbb{Z}[S] \to B$$

(evaluating at $*$). Then let $\eta(f)$ be the map

$$S \to \text{Hom}(A, B), \quad s \mapsto (a \mapsto f(a \otimes s)).$$

We need to show that this is continuous for the compact-open topology. A characterisation of the compact-open topology is the following: a map

$$g : S \to \text{Hom}(A, B)$$

is continuous for the compact-open topology if and only if the map

$$A \times S \to B, \quad (a, s) \mapsto g(s)(a)$$

is continuous. Now, the map

$$(a, s) \mapsto f(a \otimes s)$$

corresponds to the map

$$f : \underline{A} \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$$

of condensed abelian groups. Since there is a natural surjection

$$\mathbb{Z}[\underline{A} \times \underline{S}] = \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}]\underline{A} \otimes \mathbb{Z}[\underline{S}],$$

this determines and is determined by a map of condensed abelian groups

$$\mathbb{Z}[\underline{A} \times \underline{S}] \to \underline{B},$$

which determines and is determined by a map of condensed sets

$$\underline{A} \times \underline{S} \to \underline{B},$$

i.e. a continuous map $A \times S \to B$, as desired. In the above we have constructed an injective map

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \to \underline{\mathrm{Hom}(A, B)}.$$

We need to show that it is surjective. Given a profinite set S with a map

$$S \to \mathrm{Hom}(A, B)$$

that is continuous for the compact-open topology, we can reverse the steps above to get a map

$$\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$$

of condensed abelian groups. We need to show that this factors through $\underline{A} \otimes \mathbb{Z}[\underline{S}]$. Now, for any abelian group G, we have a partial resolution (i.e. exact sequence)

$$\mathbb{Z}[G \times G] \to \mathbb{Z}[G] \to G \to 0$$

where the first map sends a generator $[(g_1, g_2)]$ to $[g_1 + g_2] - [g_1] - [g_2]$. Since this holds for any abelian group, the same result is true in condensed abelian groups, so we have a partial resolution

$$\mathbb{Z}[\underline{A} \times \underline{A}] \to \mathbb{Z}[\underline{A}] \to \underline{A} \to 0.$$

We can tensor the resolution with $\mathbb{Z}[\underline{S}]$ for a profinite S and it stays exact (the tensor product is right exact). We now have an exact sequence

$$\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{A} \otimes \mathbb{Z}[\underline{S}] \to 0.$$

Further, $\mathrm{Hom}_{\mathrm{CondAb}}(-, \underline{B})$ is a left exact functor, meaning that we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}_{\mathrm{CondAb}}\left(\underline{A} \otimes \mathbb{Z}[\underline{S}], \underline{B}\right) \to \mathrm{Hom}_{\mathrm{CondAb}}\left(\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}], \underline{B}\right) \\ &\to \mathrm{Hom}_{\mathrm{CondAb}}\left(\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}], \underline{B}\right). \end{aligned}$$

By this exact sequence, showing that the map $\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$ factors through $\underline{A} \otimes \mathbb{Z}[\underline{S}]$ is equivalent to showing that the composition

$$\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$$

is zero. We have

$$\mathbb{Z}[\underline{A} \times \underline{A}] \otimes \mathbb{Z}[\underline{S}] = \mathbb{Z}[\underline{A} \times \underline{A} \times \underline{S}] = \mathbb{Z}[\underline{A \times A \times S}]$$

and similarly,

$$\mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] = \mathbb{Z}[\underline{A \times S}.$$

The composition above corresponds to a composition of continuous maps

$$A \times A \times S \to A \times S \to B$$

and due to the way in which we defined our resolution, the map on the left is

$$(a_1, a_2, s) \mapsto (a_1 + a_2 - a_1 - a_2, s) = (0, s).$$

The map on the right corresponds to a map $S \to \mathrm{Hom}(A, B)$, i.e. for fixed s it is a group homomorphism $A \to B$ and in particular takes 0 to 0. Therefore, the composition is zero as desired. □

(4.3.2) Remark. By theorem (4.1.5) and proposition (4.1.17), we can map locally compact abelian groups fully faithfully into condensed abelian groups with the functor $A \mapsto \underline{A}$. Moreover, by proposition (4.3.1), the internal hom is well-behaved with respect to this inclusion. In the following, we will discuss how to compute $R\underline{\mathrm{Hom}}$'s between condensed abelian groups associated to locally compact abelian groups.

(4.3.3) Theorem. For any condensed abelian group A, there is a resolution (i.e. exact sequence), functorial in A, of the form

$$\cdots \to \bigoplus_{j=1}^{n_p} \mathbb{Z}[A^{r_{p,j}}] \to \cdots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0$$

where $n_p, r_{p,j}$ are non-negative integers.

(4.3.4) Remark. The resolution in (4.3.3) is due to Eilenberg, Mac Lane, Breen and Deligne and we refer to it as the EMBD resolution. See [Sch19b, Theorem 4.5, Remark 4.6, and Appendix to Lecture IV].

(4.3.5) Theorem. For condensed abelian groups A, M and extremally disconnected set S, there is a spectral sequence

$$E_1^{p,q} = \prod_{j=1}^{n_p} H^q_{\mathrm{cond}}(A^{r_{p,j}} \times \underline{S}, M) \Rightarrow \underline{\mathrm{Ext}}^{p+q}(A, M)(S)$$

that is functorial in A, M, S.

Proof. We apply $R\operatorname{Hom}(-, M)$ to the EMBD resolution of $A \otimes \mathbb{Z}[\underline{S}]$, which has the form

$$\cdots \to \bigoplus_{j=1}^{n_p} \mathbb{Z}[A^{r_{p,j}} \times \underline{S}] \to \cdots \to \mathbb{Z}[A^3 \times \underline{S}] \oplus \mathbb{Z}[A^2 \times \underline{S}]$$
$$\to \mathbb{Z}[A^2 \times \underline{S}] \to \mathbb{Z}[A \times \underline{S}] \to A \otimes \mathbb{Z}[\underline{S}] \to 0,$$

and use the fact that $R\underline{\operatorname{Hom}}(A, M)(S) = R\operatorname{Hom}(A \otimes \mathbb{Z}[\underline{S}], M)$. We conclude using theorem (4.2.4). □

(4.3.6) RHoms between locally compact abelian groups. Let A and B be locally compact abelian groups. We would like to be able to calculate $R\underline{\operatorname{Hom}}(\underline{A}, \underline{B})$ and relate it to an existing notion of $R\operatorname{Hom}$ between locally compact abelian groups due to Hoffmann and Spitzweck [HS07]. Thanks to the structure theorem (4.1.11) (including Pontrjagin duality (4.1.14)), this computation can be reduced to a few cases of simple locally compact abelian groups (see [Sch19b, Corollary 4.9]) the most important two being given by theorem (4.3.7).

(4.3.7) Theorem. Consider the condensed abelian group associated to a compact abelian group consisting of a product of circles

$$A = \underline{\prod_I \mathbb{T}} = \underline{\prod_I \mathbb{R}/\mathbb{Z}}.$$

where I is any set. We have the following

(i) For any discrete condensed abelian group M (i.e. $M = \underline{M'}$ where M' is a discrete abelian group),

$$R\underline{\operatorname{Hom}}(A, M) = \bigoplus_I M[-1]$$

where the isomorphism

$$\bigoplus_I M[-1] \to \underline{R\operatorname{Hom}(A, M)}$$

is induced by the maps

$$M[-1] = R\underline{\operatorname{Hom}}(\underline{\mathbb{Z}}[1], M) \to R\underline{\operatorname{Hom}}(\underline{\mathbb{R}/\mathbb{Z}}, M) \to R\underline{\operatorname{Hom}}(A, M),$$

where the last map is induced from the projection $p_i : \prod \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to the i-th factor, $i \in I$.

(ii)

$$R\underline{\operatorname{Hom}}(A, \mathbb{R}) = 0$$

Proof of (i). We start with the case where I is finite. In fact, this reduces to I having one element, as the product becomes a direct sum which we pull out of the $R\underline{\mathrm{Hom}}$ (it commutes with direct sums in the first variable, being a right adjoint). So we need to show that the map

$$M[-1] = R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \to R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, M)$$

is an isomorphism. Now we have the distinguished triangle $(\mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z})$ and by (TR2), this means that the triangle $(\mathbb{R}, \mathbb{R}/\mathbb{Z}, \mathbb{Z}[1])$ is ditinguished. Thus, since $R\underline{\mathrm{Hom}}(-, M)$ is a triangulated functor, we have a distinguished triangle

$$R\underline{\mathrm{Hom}}(\mathbb{Z}[1], M) \to R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, M) \to R\underline{\mathrm{Hom}}(\mathbb{R}, M)$$

so to show that the morphism between the first two is an isomorphism in the derived category, we need to show that the third term is zero,

$$R\underline{\mathrm{Hom}}(\mathbb{R}, M) = 0 = R\underline{\mathrm{Hom}}(0, M).$$

To prove this, we use the spectral sequence in (4.3.5) and we see that it suffices to show that the map

$$H^q_{\mathrm{cond}}(\underline{\mathbb{R}^r} \times \underline{S}, M) \to H^q_{\mathrm{cond}}(S, M),$$

induced from the map $(0, \mathrm{id}_S) : S \to \mathbb{R}^r \times S$, is an isomorphism for all q and r. Using the comparison with classical cohomology of compact Hausdorff spaces in theorem (3.2.5) and noting that $[-n, n]^r \times S$ is homotopy equivalent to S for all $n \in \mathbb{N}$, we have an isomorphism

$$H^q_{\mathrm{cond}}(\underline{[-n, n]^r} \times \underline{S}, M) \to H^q_{\mathrm{cond}}(S, M),$$

induced by the same map. By lemma (2.7.6), the complex calculating the right hand side is

$$R\Gamma(S, M) = R \varprojlim_n R\Gamma(S, M)$$

so it suffices to show that

$$R \varprojlim_n R\Gamma([-n, n]^r \times S, M)$$

calculates the cohomology

$$H^q_{\mathrm{cond}}(\underline{\mathbb{R}^r} \times \underline{S}, M).$$

In other words, we need to show that

$$R\,\mathrm{Hom}(\mathbb{Z}[\underline{\mathbb{R}^r \times S}], M) = R \varprojlim_n R\,\mathrm{Hom}(\mathbb{Z}[\underline{[-n, n]^r \times S}], M).$$

But since $R\,\mathrm{Hom}(-, M)$ is a triangulated functor, it suffices to show that

$$\mathbb{Z}[\underline{\mathbb{R}^r \times S}] = \mathrm{hocolim}_n\, \mathbb{Z}[\underline{[-n, n]^r \times S}]$$

but this follows from the fact that $\mathbb{R}^r \times S$ is the increasing union (i.e. filtered colimit over $\mathbb{N}$) of the spaces $[-n, n]^r \times S$, lemma (2.7.7) and the passage from topological spaces to condensed abelian groups.

Passing to the case of a possibly infinite indexing set I, it is enough to show that the map

$$\varinjlim_{J \subset I} R\underline{\mathrm{Hom}}\left(\underline{\prod_J \mathbb{T}, M}\right) \to R\underline{\mathrm{Hom}}\left(\underline{\prod_I \mathbb{T}, M}\right),$$

where the colimit is taken over all finite subsets J of I (and is thus filtered), is an isomorphism. The spectral sequence in (4.3.5) reduces this to proving that

$$\varinjlim_{J \subset I} H^q_{\mathrm{cond}}\left(\underline{\prod_J \mathbb{T}^r, M}\right) \to H^q_{\mathrm{cond}}\left(\underline{\prod_I \mathbb{T}^r, M}\right),$$

is an isomorphism. Here we are working with cohomology on compact Hausdorff spaces and thus theorem (3.2.5) applies, and we can use this known result (see [ES52, Chapter X, Theorem 3.1]) for sheaf/Čech cohomology.

For the proof of (ii), we refer to [Sch19b, Proof of Theorem 4.3]. □

Appendix A

Categories

We fix here some conventions about categories, most importantly about limits and colimits, as the terminology for these has varied quite a lot throughout history. For proofs and more details we refer to the literature, for example [Mac98] and [Sta21],

A.1 Limits and colimits

(A.1.1) Definition. Let $I \to \mathcal{C}$ be a functor, for which the image of an object $i \in I$ is denoted X_i, and the image of a morphism $(\varphi : i \to j)$ is denoted φ_{ij}.

- The *colimit* of the functor $I \to \mathcal{C}$ is an object $\varinjlim_i X_i$ equipped with morphisms $f_j : X_j \to \varinjlim_i X_i$ such that for any morphism $\varphi : j \to k$ in I, we have $f_j = f_k \circ \varphi_{jk}$. Moreover, we require the colimit to be universal with respect to this property, i.e. for every object C equipped with morphisms $c_i : X_i \to C$ such that $c_i = c_j \circ \varphi_{ij}$ for all i, j, φ, there is a unique morphism $c : \operatorname{colim}_i X_i \to C$ such that $c \circ f_i = c_i$ for all i.
- The *limit* of the functor $I \to \mathcal{C}$ is an object $\varprojlim_i X_i$ equipped with morphisms $f_j : \varprojlim_i X_i \to X_j$ such that for any morphism $\varphi : j \to k$ in I, we have $f_k = \varphi_{jk} \circ f_j$. Moreover, we require the limit to be universal with respect to this property, i.e. for every object P equipped with morphisms $p_i : P \to X_i$ such that $p_j = \varphi_{ij} \circ p_i$ for all i, j, φ, there is a unique morphism $p : P \to \varprojlim_i X_i$ such that $f_i \circ p = p_i$ for all i.

(A.1.2) Definition.

- A category I is *filtered* if it is nonempty, for any pair of objects i, i', there is an object k and arrows $i \to k$ and $i' \to k$, and for any pair of morphisms $f, g : i \to i'$ there is a j and an morphism $h : i' \to j$ such that $h \circ g = h \circ f$.

- A category I is *cofiltered* if it is nonempty, for any pair of objects i, i', there is an object k with arrows $k \to i$ and $k \to i'$, and for any pair of morphisms $f, g : i \to i'$ there is an object j and morphism $h : j \to i$ such that $f \circ h = g \circ h$ (in other words, its opposite category is filtered).

(A.1.3) Definition. A *filtered colimit* (resp. *cofiltered limit*) is a the colimit (resp. limit) of a functor $I \to \mathcal{C}$ where I is a filtered (resp. cofiltered) category.

(A.1.4) Definition. A partially ordered set I is called *directed* if it is nonempty and for all $i, j \in I$, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

(A.1.5) Definition.

- Let I be a directed partially ordered set regarded as a category in the natural way. Then a functor $I \to \mathcal{C}$ is called a *directed system* in $\mathcal{C}$.

- Let I be a directed partially ordered set. Then a functor $I^{\mathrm{op}} \to \mathcal{C}$ is called an *directed inverse system* in $\mathcal{C}$.

(A.1.6) Remark.

- A directed system in $\mathcal{C}$ is given by a collection of objects $(M_i)_i$ in $\mathcal{C}$ and for each $i \leq j$ an arrow $\varphi_{ij} : M_i \to M_j$ such that $\varphi_{ii} = \mathrm{id}_{M_i}$ for all i and $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all $i \leq j \leq k$.

- A directed inverse system in $\mathcal{C}$ is given by a collection of objects $(M_i)_i$ in $\mathcal{C}$ and for each $i \leq j$ an arrow $\varphi_{ij} : M_j \to M_i$ such that $\varphi_{ii} = \mathrm{id}_{M_i}$ for all i and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ for all $i \leq j \leq k$.

(A.1.7) Proposition. ([Sta21, Lemma 0032]) Every filtered colimit in $\mathcal{C}$ is isomorphic to a colimit of a directed system and every cofiltered limit in $\mathcal{C}$ is isomorphic to a limit of a directed inverse system.

(A.1.8) Definition. Suppose all finite limits (resp. finite colimits) exist in the category $\mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *left exact* (resp. *right exact*) if it commutes with all finite limits (resp. colimits). It is called *exact* if it is both left and right exact.

A.2 Adjoints

(A.2.1) Definition. A pair (F, G) where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors is called an *adjoint pair of functors* (F *is left adjoint to* G*;* G *is right adjoint to* F) if for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$ there are isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(Y)),$$

natural in X, Y.

(A.2.2) Remark. Taking $Y = F(X)$ we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(F(X), F(X)) \to \mathrm{Hom}_{\mathcal{C}}(X, G(F(X)))$$

and in particular a morphism $\eta_X : X \to G(F(X))$ corresponding to $\mathrm{Id}_{F(X)}$. Similarily for $X = G(Y)$ we have

$$\mathrm{Hom}_{\mathcal{D}}(F(G(Y)), Y) \to \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y))$$

and again a morphism $\varepsilon_Y : F(G(Y)) \to Y$ corresponding to $\mathrm{Id}_{G(Y)}$.

These give morphisms of functors

$$\eta : \mathrm{Id}_{\mathcal{C}} \to G \circ F \quad \text{(unit)}$$

and

$$\varepsilon : F \circ G \to \mathrm{Id}_{\mathcal{D}} \quad \text{(counit)}$$

(A.2.3) Theorem. (Freyd, [Mac98, Theorem 2, p. 121]) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and suppose $\mathcal{C}$ has all (co)limits. Under a solution set condition which is satisfied for all categories considered in this thesis, the following holds: F has a left (right) adjoint if and only if it preserves (co)limits. □

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